# Tychonoff Expansions with Prescribed Resolvability Properties

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**Abstract**

The recent literature offers examples, specific and hand-crafted, of Tychonoff spaces (in ZFC) which respond negatively to these questions, due respectively to Ceder and Pearson (1967) and to Comfort and García-Ferreira (2001): (1) Is every $\omega$-resolvable space maximally resolvable? (2) Is every maximally resolvable space extraresolvable? Now using the method of $\mathcal{KID}$ expansion, the authors show that *every* suitably restricted Tychonoff topological space $(X, \mathcal{T})$ admits a larger Tychonoff topology (that is, an"expansion") witnessing such failure. Specifically the authors show in ZFC that if $(X, \mathcal{T})$ is a maximally resolvable Tychonoff space with $S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T}) = \kappa$, then $(X, \mathcal{T})$ has Tychonoff expansions $\mathcal{U} = \mathcal{U}_i$ $(1 \leq i \leq 5)$, with $\Delta(X,\mathcal{U}_i) = \Delta(X, \mathcal{T})$ and $S(X,\mathcal{U}_i) \leq \Delta(X,\mathcal{U}_i)$, such that $(X,\mathcal{U}_i)$ is: $(i = 1)$ $\omega$-resolvable but not maximally resolvable; $(i = 2)$ [if $\kappa'$ is regular, with $S(X, \mathcal{T}) \leq \kappa' \leq \kappa$] $\tau$-resolvable for all $\tau < \kappa'$, but not $\kappa'$-resolvable; $(i = 3)$ maximally resolvable, but not extraresolvable; $(i = 4)$ extraresolvable, but not maximally resolvable; $(i = 5)$ maximally resolvable and extraresolvable, but not strongly extraresolvable.

*Keyword: Resolvable space, extraresolvable space, strongly extraresolvable space, maximally resolvable space, $\omega$-resolvable space, Souslin number, independent family*

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## 1 Introduction, Definitions and Notation

Our principal interest is in Tychonoff spaces, i.e., in completely regular, Hausdorff spaces, and all spaces $(X, \mathcal{T})$ hypothesized here, also all expansions (refinements) of $\mathcal{T}$ constructed, will be Tychonoff topologies. The topological properties we consider, however, are intelligible (a wonderful word in this context, borrowed from Hewitt [20]) for arbitrary spaces, so in 1.2 below, which defines the properties we consider, we impose no separation hypotheses.

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**Notation 1.1** For $X$ a set and $\tau$ a cardinal, we set $[X]^\tau := \{A \subseteq X : |A| = \tau\}$. The symbols $[X]^{<\tau}$ and $[X]^{\leq\tau}$ are defined analogously.

The symbol $D(\tau)$ denotes the discrete space of cardinality $\tau$.

When $X = (X, \mathcal{T})$ is a space and $Y \subseteq X$, we denote by $(Y, \mathcal{T})$ the set $Y$ with the subspace topology inherited from $X$.

The symbols $w$ and $d$ denote *weight* and *density character*, respectively. For a space $X = (X, \mathcal{T})$, the *dispersion character* $\Delta(X)$ is the smallest cardinal of an nonempty open subset of $X$, and $\mathrm{nwd}(X)$, the *nowhere density number of* $X$, is

$\mathrm{nwd}(X) := \min\{|A| : A \subseteq X, \mathrm{int}_X \, \mathrm{cl}_X \, A \neq \emptyset\}$.

Evidently $\mathrm{nwd}(X)$ coincides with the *open density number of* $X$ [6] defined by

$\mathrm{od}(X) := \min\{d(U) : \emptyset \neq U \in \mathcal{T}\}$,

which has also been denoted $d_0(X)$ [26].

As in [8] and [25], a subset $D$ of a space $X = (X, \mathcal{T})$ is $\tau$-*dense* in $X$ if $|D \cap U| \geq \tau$ whenever $\emptyset \neq U \in \mathcal{T}$. It is obvious that if $D$ is dense in a space $X$ with $\mathrm{nwd}(X) \geq \tau$, then $D$ is $\tau$-dense in $X$.

$(X, \mathcal{T})$ is *crowded* if no point of $X$ is isolated in the topology $\mathcal{T}$. (This term, introduced by van Douwen [13], has been adopted subsequently by many authors [14], [22], [25]. Others have called such a space *dense-in-itself* [7].)

A family of nonempty pairwise disjoint open subsets of $X = (X, \mathcal{T})$ is a *cellular family*, and $S(X)$, the *Souslin number* of $X$, is

$S(X) := \min\{\kappa : \text{no cellular } \mathcal{U} \subseteq \mathcal{T} \text{ satisfies } |\mathcal{U}| = \kappa\}$.

**Definition 1.2** Let $X = (X, \mathcal{T})$ be a space. Then $X$ is

(i) *resolvable* (Hewitt [20]) if it has two complementary dense subsets;

(ii) $\kappa$-*resolvable* (Ceder [2]) if there is a family of $\kappa$-many pairwise disjoint dense subsets of $X$;

(iii) *maximally resolvable* (Ceder [2]) if it is $\Delta(X)$-resolvable;

(iv) *extraresolvable* (Malykhin [28]) if there is a family $\mathcal{D}$ of dense subsets, with $|\mathcal{D}| \geq (\Delta(X))^+$, such that every two elements of $\mathcal{D}$ have interesection which is nowhere dense in $X$; and

(v) *strongly extraresolvable* (Comfort and García-Ferreira [4], [5]) if there is a family $\mathcal{D}$ of dense subsets, with $|\mathcal{D}| \geq (\Delta(X))^+$, such that distinct $D_0$, $D_1 \in \mathcal{D}$ satisfy $|D_0 \cap D_1| < \mathrm{nwd}(X)$.

**Remark 1.3** In early versions of this manuscript, circulated privately to selected colleagues, we were able to establish item ($i = 4$) of the Abstract, even its special case Theorem 3.9, only under the additional assumption that there exists a cardinal $\tau$ such that $\tau < \kappa < 2^\tau$. Indeed, although we had shown in [8] the existence of extraresolvable Tychonoff spaces which are not maximally resolvable when GCH fails, it was an unsolved problem whether such spaces exist in ZFC. That question has been settled affirmatively by Juhász, Shelah and Soukup [27]. We are grateful to those authors for furnishing us with a pre-publication copy of their work.

**Definition 1.4** Let $\kappa \geq \omega$.

(a) A partition $\mathcal{B}$ of $\kappa$ is a $\kappa$-partition if each $B \in \mathcal{B}$ satisfies $|B| = \kappa$;

(b) a family $\mathcal{B} = \{\mathcal{B}_t : t \in T\}$ of partitions $\mathcal{B} = \{\mathcal{B}_t^\alpha : \alpha < \kappa_t\}$ of $\kappa$ is $\tau$-*independent* (with $1 \leq \tau \leq \kappa$) if $|\bigcap_{t\in F} B_t^{f(t)}| \geq \tau$ for each $F \in [T]^{<\omega}$ and $f \in \Pi_{t\in F}\,\kappa_t$.

(c) a family $\mathcal{B} = \{\mathcal{B}_t : t \in T\}$ of indexed partitions $\mathcal{B}_t = \{B_t^\alpha : \alpha < \kappa_t\}$ (with $2 \leq \kappa_t \leq \kappa$ for each $t \in T$) *separates points* [resp., *separates small sets*] if for distinct $x, x' \in \kappa$ there are $\mathcal{B}_t \in \mathcal{B}$ and (distinct) $\alpha, \alpha' < \kappa_t$ such that $x \in B_t^\alpha$ and $x' \in B_t^{\alpha'}$ [resp., for disjoint $S, S' \in [\kappa]^{<\kappa}$ there are $\mathcal{B}_t \in \mathcal{B}$ and (distinct) $\alpha, \alpha' < \kappa_t$ such that $S \subseteq B_t^\alpha$ and $S' \subseteq B_t^{\alpha'}$].

It is obvious that any partition in a $\kappa$-independent family (of partitions of $\kappa$) is necessarily a $\kappa$-partition.

**Discussion 1.5** Given a point-separating family $\mathcal{B}$ as in Definition 1.4, we denote by $\mathcal{T}_\mathcal{B}$ the smallest topology on $\kappa$ in which each set $B_t^\alpha \in \mathcal{B}_t \in \mathcal{B}$ is open; clearly each such $B_t^\alpha$ is $\mathcal{T}_\mathcal{B}$-closed, and $\{\bigcap_{t\in F} B_t^{f(t)} : F \in [T]^{<\omega}, f \in \Pi_{t\in F}\,\kappa_t\}$ is a basis for $\mathcal{T}_\mathcal{B}$. (This is a Hausdorff topology since $\mathcal{B}$ separates points of $\kappa$, hence is a Tychonoff topology since it has a clopen basis.) The evaluation map $e_\mathcal{B} : (\kappa, \mathcal{T}_\mathcal{B}) \to \Pi_{t\in T}\,D(\kappa_t)$ given by

$$(e_\mathcal{B} x)_t = \alpha \textit{ if } x \in B_t^\alpha \qquad (x \in \kappa, t \in T, \alpha < \kappa_t)$$

is a homeomorphism from $(\kappa, \mathcal{T}_\mathcal{B})$ onto a subspace $X$ of the Tychonoff space $K := \Pi_{t\in T}\,D(\kappa_t)$. That $X := e_\mathcal{B}[\kappa]$ is dense in $K$ follows from the fact that $\mathcal{B}$ is 1-independent. Conversely, given $K = \Pi_{t\in T}\,D(\kappa_t)$ with $|T| = 2^\kappa$ and with $2 \leq \kappa_t \leq \kappa$ for each $t \in T$, the Hewitt-Marczewski-Pondiczery theorem (cf. [16](2.3.15), [11](§3 and Notes)) gives a dense set $X \subseteq K$ such that $|X| = \kappa$, and then the family $\mathcal{B} := \{\mathcal{B}_t : t \in T\}$ with $\mathcal{B}_t := \{\pi_t^{-1}(\{\alpha\} \cap X : \alpha < \kappa_t\}$ is a 1-independent family of partitions of $\kappa$ (the set $\kappa$ here being identified with the subspace $X$ of $K$). One may ensure that each $\mathcal{B}_t \in \mathcal{B}$ is a $\kappa$-partition by the following device (here we argue much as in [7](3.8) and [8](1.5)): Give each space $D(\kappa_t)$ the structure of a topological group, so that $K$ is a topological group, let $X^*$ be dense in $K$ with $|X^*| = \kappa$, and with $\langle X^*\rangle$ the subgroup of $K$ generated by $X^*$ let $X$ be the union of $\kappa$-many cosets of $\langle X^*\rangle$ in $K$. Then $B_t^\alpha := \pi_t^{-1}(\{\alpha\}) \cap X$ satisfies $|B_t^\alpha| = \kappa$ for each $\alpha < \kappa_t$, $t \in T$; indeed more generally each basic open set $U$ in $X$ (of the form $U = (\bigcap_{i=1}^n \pi_{t_i}^{-1}(\{\alpha_i\})) \cap X$, with $\alpha_i < \kappa_{t_i}$, $n < \omega$) satisfies $|U| = \kappa$, so the family $\mathcal{B}$ is even $\kappa$-independent, and $\Delta(X) = \kappa$.

The correspondence $\mathcal{B} \leftrightarrow X$ just described is of Galois type in the sense that when dense $X \subseteq K = \Pi_{t\in T}\,D(\kappa_t)$ is given with $|X| = \kappa$ and the family $\mathcal{B} = \{\mathcal{B}_t : t \in T\}$ is defined, then $e_\mathcal{B} : (\kappa, \mathcal{T}_\mathcal{B}) \to K$ satisfies $e_\mathcal{B}[\kappa] = X$.

In this paper in this context, $T$ and $\{\kappa_t : t \in T\}$ being given, we use the notations $(\kappa, \mathcal{T}_\mathcal{B})$, $(X, \mathcal{T}_\mathcal{B})$ and $e_\mathcal{B}[\kappa]$ interchangeably.

The point-separating family described in Discussion 1.5 may be chosen to separate small sets in a strong sense. Lemma 1.6, which exploits a trick introduced by Eckertson [14] in a related context, strengthens a statement given in our works [6] and [7](3.3(b)). When reference is made, in Lemma 1.6 and later, to a partition $\{T(\lambda) : \lambda \in \Lambda\}$ of $T$, the trivial (one-cell) partition is not excluded.

**Lemma 1.6** *Let $\kappa \geq \omega$ and $|T| = 2^\kappa$, and for $t \in T$ let $2 \leq \kappa_t \leq \kappa$. Let $\{T(\lambda) : \lambda \in \Lambda\}$ be a partition of $T$, with each $|T(\lambda)| = 2^\kappa$. Then there is a $\kappa$-independent family $\mathcal{I} = \{\mathcal{I}_t : t \in T\}$ of partitions of $\kappa$, with $|\mathcal{I}_t| = \kappa_t$ for each $t \in T$, such that for every ordered pair $(S, S')$ of disjoint elements of $[\kappa]^{<\kappa}$ and for every $\lambda \in \Lambda$ there are infinitely many $t \in T(\lambda)$ such that $S \subseteq I_t^0$ and $S' \subseteq I_t^1$.*

Proof. Let $\mathcal{B} = \{\mathcal{B}_t : t \in T\}$ be a point-separating $\kappa$-independent family of partitions of $\kappa$ with $|T| = 2^\kappa$ and with $|\mathcal{B}_t| = \kappa_t$ for each $t \in T$, as given in Discussion 1.5. For $\lambda \in \Lambda$ let $\{T(\lambda, \xi) : \xi < 2^\kappa\}$ be a partition of $T(\lambda)$ with each $|T(\lambda, \xi)| = \omega$, and using $|[\kappa]^{<\kappa}| \leq 2^\kappa$ let $\{(S_\xi, S'_\xi) : \xi < 2^\kappa\}$ list all ordered pairs of disjoint members of $[\kappa]^{<\kappa}$ (with repetitions permitted). Then define $\mathcal{I} = \{\mathcal{I}_t : t \in T\}$ with $\mathcal{I}_t = \{I_t^\alpha : \alpha < \kappa_t\}$ as follows: if $t \in T(\lambda, \xi)$, then

$I_t^0 = (B_t^0 \cup S_\xi) \backslash S'_\xi$, $I_t^1 = (B_t^1 \cup S'_\xi) \backslash S_\xi$, and $I_t^\alpha = B_t^\alpha \backslash (S_\xi \cup S'_\xi)$ for $2 \leq \alpha < \kappa_t$.

Then each $\mathcal{I}_t$ is a partition of $\kappa$, and since

$$B_t^\alpha \triangle I_t^\alpha \in [\kappa]^{<\kappa} \qquad (*)$$

for each $t \in T$ and $\alpha < \kappa_t$ with $\mathcal{B}_t$ a $\kappa$-partition, so also is each $\mathcal{I}_t$ a $\kappa$-partition. Further for each pair $(S, S') = (S_\xi, S'_\xi)$ we have $S \subseteq I_t^0$ and $S' \subseteq I_t^1$ for each $t \in T(\lambda, \xi) \in [T(\lambda)]^\omega$, as required. $\square$

**Definition 1.7** With $\{\kappa_t : t \in T\}$ and $\{T(\lambda) : \lambda \in \Lambda\}$ given as in Lemma 1.6, a $\kappa$-independent family $\mathcal{I}$ of partitions of $\kappa$ with the additional property given there is a *strong small-set-separating* family of partitions which *respects* the partition $\{T(\lambda) : \lambda \in \Lambda\}$ of $T$.

**Remark 1.8** Clearly a $\kappa$-independent family $\{\mathcal{I}_t : t \in T\}$ of partitions of $\kappa$, if it respects some partition $\{T(\lambda) : \lambda \in \Lambda\}$ of $T$, also respects the trivial (one-cell) partition. Usually in this paper we apply Lemma 1.6 only in the context of the trivial partition; in what follows, if no explicit reference is made to the partition which a strong small-set-separating family of $\kappa$-partitions respects, we intend by default the trivial partition.

The following theorem augments, simplifies and extends arguments given in our works [7](3.8) and [8](1.6). As usual when a point-separating family $\mathcal{I}$ of partitions of $\kappa$ is given, we do not distinguish notationally between $\kappa$ and the space $X := e_{\mathcal{I}}[\kappa] \subseteq K = \Pi_{t \in T} D(\kappa_t)$, nor between a set $I_t^\alpha \in \mathcal{I}_t \in \mathcal{I}$ and its image $e_{\mathcal{I}}[I_t^\alpha]$ in $X$.

**Theorem 1.9** *Let $\kappa \geq \omega$ and $|T| = 2^\kappa$, and for $t \in T$ let $2 \leq \kappa_t \leq \kappa$. Then there is a $\kappa$-independent family $\mathcal{I} = \{\mathcal{I}_t : t \in T\}$ of partitions of $\kappa$ with the strong small-set-separating property, and with $|\mathcal{I}_t| = \kappa_t$ for each $t \in T$, such that the space*

$$X := e_{\mathcal{I}}[\kappa] \subseteq K := \Pi_{t \in T} D(\kappa_t)$$

*has these properties:*

(a) *$X$ is dense in $K$;*
(b) *$X$ is $\kappa$-resolvable;*
(c) *$|X| = \Delta(X) = \mathrm{nwd}(X) = \kappa$; and*
(d) *each $S \in [X]^{<\kappa}$ is closed and discrete in $X$.*

Proof. Let $\overline{T} := T \cup \{\overline{t}\}$ with $\overline{t} \notin T$, and set $\kappa_{\overline{t}} := \kappa$. Apply Lemma 1.6 with $\{\overline{T}\}$ the one-cell partition of $\overline{T}$: There is a $\kappa$-independent family $\overline{\mathcal{I}} = \{\mathcal{I}_t : t \in \overline{T}\}$ of $\kappa$-partitions of $\kappa$ with the strong small-set-separating property, with $|\mathcal{I}_t| = \kappa_t$ for each $t \in \overline{T}$ (in particular, with $|\mathcal{I}_{\overline{t}}| = \kappa_{\overline{t}} = \kappa$). By the argument given in Discussion 1.5 the set $X := e_{\mathcal{I}}[\kappa]$ is dense in $K := \Pi_{t \in T}\, D(\kappa_t)$, so (a) is proved. For $F \in [T]^{<\omega}$ and $f \in \Pi_{t \in F}\, \kappa_t$ and each $I^{\alpha}_{\overline{t}}$ (with $\alpha < \kappa_{\overline{t}} = \kappa$) we have

$$|(\bigcap_{t \in F} I^{f(t)}_t) \cap I^{\alpha}_{\overline{t}}| = \kappa \qquad (*)$$

since the family $\overline{\mathcal{I}}$ is $\kappa$-independent. Relation (*) shows that each set $e_{\mathcal{I}}[I^{\alpha}_{\overline{t}}]$ is dense in $X$ (thus proving (b)), and it shows also that $|X| = \Delta(X) = \kappa$.

Since $X$ is a crowded space, every closed, discrete subspace of $X$ is nowhere dense; so the relation $\mathrm{nwd}(X) = \kappa$ will follow from (d). Given $S \in [\kappa]^{<\kappa}$ and $x \in \kappa \backslash S$, there is $t \in T$ such that $x \in I^0_t$ and $S \subseteq I^1_t$; since $I^0_t$ and $I^1_t$ are disjoint and clopen in $X$, we conclude that $S$ is closed. Similarly if $x \in S \in [\kappa]^{<\kappa}$ there is $t \in T$ such that $x \in I^0_t$ and $S\backslash\{x\} \subseteq I^1_t$, so $I^0_t \cap S = \{x\}$; it follows that $S$ is discrete. □

**Remarks 1.10** (a) In earlier work [8] by a different argument we have demonstrated the existence of a $\kappa$-resolvable dense subset $X$ of some spaces of the form $\Pi_{t \in T}\, D(\kappa_t)$ with $|T| = 2^{\kappa}$, even with $|X| = \Delta(X) = \mathrm{nwd}(X) = \kappa$. (See also [6](5.3 and 5.4) for similar results.) The argument of Theorem 1.9 is preferable, both because of its simplicity and because it gives in concrete form a family $\mathcal{I}$ for which $X = e_{\mathcal{I}}[\kappa]$; this latter feature is essential in the proof of Lemma 3.7 below.

(b) The case in Definition 1.4 in which there is $\lambda \in [2, \kappa]$ such that $\kappa_t = \lambda$ for all $t \in T$, together with passage in that case from $\mathcal{B}$ to the space $(\kappa, \mathcal{T}_{\mathcal{B}}) = (X, \mathcal{T}_{\mathcal{B}})$, has been used by many authors in connection with resolvability questions [13], [6], [7], [25], [8].

## 2 The $\mathcal{KID}$ Expansion: Transfer from $\mathcal{T}$ to $\mathcal{T}_{\mathcal{KID}}$

Here we explain and develop further the techniques originating in [21], [22]. In broad terms the goal, given a crowded Tychonoff space $(X, \mathcal{T})$, is to augment ("expand") the topology $\mathcal{T}$ to a larger crowded Tychonoff topology $\mathcal{T}_{\mathcal{KID}}$ in such a way that certain specified $\mathcal{T}$-dense subsets of $X$ remain $\mathcal{T}_{\mathcal{KID}}$-dense, while certain other subsets of $X$ become closed and discrete in the topology $\mathcal{T}_{\mathcal{KID}}$.

In Definition 2.2, the transition from $\mathcal{T}$ to the $\mathcal{T}_{\mathcal{KID}}$-open sets $W^{\alpha}_t$ is effected *via* the intermediate sets $H^{\alpha}_t$. Their definition depends on the hypothesized dense array $\mathcal{D}$ and the $\kappa$-independent family $\mathcal{I}$, but not on the family $\mathcal{K}$.

The following notation is as in [7](3.2).

**Notation 2.1** Let $X$ be a set with $|X| = \kappa \geq \omega$, and let $\mathcal{D} = \{D^{\gamma}_{\eta} : \gamma < \tau, \eta < \kappa\}$ be a partition of $X$ with $1 \leq \tau \leq \kappa$. Then for $S \subseteq \kappa$ the set $X(S) \subseteq X$ is defined by

$$X(S) := \bigcup\{D^{\gamma}_{\eta} : \gamma < \tau, \eta \in S\}.$$

**Definition 2.2** Let $(X, \mathcal{T})$ be a crowded Tychonoff space with $|X| = \kappa \geq \omega$, fix nonempty $Z \subseteq X$, and let $\mathcal{I} = \{\mathcal{I}_t : t \in Z \times 2^{\kappa}\}$ be a point-separating $\kappa$-independent family of partitions of $\kappa$ with $\mathcal{I}_t = \{I^{\alpha}_t : \alpha < \kappa_t\}$, $2 \leq \kappa_t \leq \kappa$ for each $t \in Z \times 2^{\kappa}$. Let $1 \leq \tau \leq \kappa$ and $\mathcal{D} = \{D^{\gamma}_{\eta} : \gamma < \tau, \eta < \kappa\}$ be a partition of $X$, and for $t \in Z \times 2^{\kappa}$ and $\alpha < \kappa_t$ set

$H_t^\alpha := X(I_t^\alpha) = \bigcup\{D_\eta^\gamma : \gamma < \tau, \eta \in I_t^\alpha\}$.

Let $\mathcal{K} = \{K_\xi : \xi < 2^\kappa\} \subseteq \mathcal{P}(Z)$, and for $t = (x, \xi) \in Z \times 2^\kappa$ and $\alpha < \kappa_t$ define $W_t^\alpha$ as follows:

If $K_\xi = \emptyset$, then $W_t^\alpha = H_t^\alpha$.

If $K_\xi \neq \emptyset$, then

$W_t^0 = (H_t^0 \cup K_\xi)\backslash\{x\}$,

$W_t^1 = (H_t^1 \backslash K_\xi) \cup \{x\}$, and

$W_t^\alpha = H_t^\alpha \backslash (K_\xi \cup \{x\})$ for $2 \leq \alpha < \kappa_t$.

For each $t \in Z \times 2^\kappa$ set

$\mathcal{H}_t := \{H_t^\alpha : \alpha < \kappa_t\}$ and $\mathcal{W}_t := \{W_t^\alpha : \alpha < \kappa_t\}$,

and set

$\mathcal{H} := \{\mathcal{H}_t : t \in Z \times 2^\kappa\}$, and $\mathcal{W} := \{\mathcal{W}_t : t \in Z \times 2^\kappa\}$.

Then

$\mathcal{T}^{\mathcal{ID}}$ is the smallest topology on $X$ such that $\mathcal{T} \subseteq \mathcal{T}^{\mathcal{ID}}$ and each $\mathcal{H}_t \subseteq \mathcal{T}^{\mathcal{ID}}$, and

$\mathcal{T}_{\mathcal{KID}}$, the $\mathcal{KID}$ *expansion* of $\mathcal{T}$, is the smallest topology on $X$ such that $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{KID}}$ and each $\mathcal{W}_t \subseteq \mathcal{T}_{\mathcal{KID}}$.

**Remarks 2.3** (a) The indexings $\mathcal{D} = \{D_\eta^\gamma : \gamma < \tau, \eta < \kappa\}$ and $\mathcal{I} = \{\mathcal{I}_t : t \in Z \times 2^\kappa\}$ in Definition 2.2 are faithful. No such restriction is imposed on the indexing $\mathcal{K} = \{K_\xi : \xi < 2^\kappa\}$. Indeed in some of the applications we will have $K_\xi = \emptyset$ for many $\xi < 2^\kappa$.

(b) For $t \in Z \times 2^\kappa$ the family $\mathcal{H}_t$ is a partition of $X$ into $\mathcal{T}^{\mathcal{ID}}$-open subsets, so each $H_t^\alpha$ is $\mathcal{T}^{\mathcal{ID}}$-clopen. Similarly, since for $t \in Z \times 2^\kappa$ the family $\mathcal{W}_t$ is a partition of $X$ into $\mathcal{T}_{\mathcal{KID}}$-open sets, also each $W_t^\alpha$ is $\mathcal{T}_{\mathcal{KID}}$-clopen. It then follows, as is required of every topology hypothesized or constructed in this paper, that:

(c) Each space of the form $(X, \mathcal{T}^{\mathcal{ID}})$, and each space of the form $(X, \mathcal{T}_{\mathcal{KID}})$, is a Tychonoff space.

(d) the topology $\mathcal{T}_{\mathcal{KID}}$ depends not only on the families $\mathcal{K}$, $\mathcal{I}$, and $\mathcal{D}$, but also on the choice of the nonempty set $Z \subseteq X$. Our notation does not reflect that fact. No confusion with ensue, indeed in (nearly) all the applications we take $Z = X$. Briefly in Theorem 3.8 we will invoke the general theory in the special case $|Z| = 1$.

To avoid irrelevancies we gave Definition 2.2 in uncluttered language, but in fact we will use the expansion $\mathcal{T}_{\mathcal{KID}}$ only when the following additional conditions are satisfied. Except when noted otherwise, we assume these henceforth throughout this Section. Furthermore when families $\mathcal{I}$, $\mathcal{D}$ and $\mathcal{K}$ have been constructed or hypothesized and $I_t^\alpha \in \mathcal{I}_t \in \mathcal{I}$, it is understood that the sets $H_t^\alpha$ and $W_t^\alpha$ are defined as in Definition 2.2.

**Standing Hypotheses and Notation 2.4**

(1) $|X| = \Delta(X, \mathcal{T}) = \kappa$;

(2) the indexed family $\mathcal{D}$ is a dense partition of $(X, \mathcal{T})$, and $D^\gamma := \bigcup_{\eta<\kappa} D_\eta^\gamma$ for $\gamma < \tau$;

(3) the family $\mathcal{I} = \{\mathcal{I}_t : t \in Z \times 2^\kappa\}$ has the strong small-set-separating property;

(4) if $F \in [2^\kappa]^{<\omega}$ then $\bigcup_{\xi \in F} K_\xi \in \mathcal{K}$; and

(5) $\xi < 2^\kappa$, $\gamma < \tau \Rightarrow \mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{ID}})}(K_\xi \cap D^\gamma) = \emptyset$.

**Lemma 2.5** [With the conventions of 2.2 and 2.4.]
*Fix $\gamma < \tau$ and $\xi < 2^\kappa$. Then*

(a) *$K_\xi$ is closed in $(Z, \mathcal{T}_{\mathcal{KID}})$;*

(b) *$(K_\xi, \mathcal{T}_{\mathcal{KID}})$ is discrete; and*

(c) *if $\emptyset \neq U \in \mathcal{T}$, $H = \bigcap_{t\in F} H_t^{f(t)}$ and $W = \bigcap_{t\in F} W_t^{f(t)}$ with $F \in [Z \times 2^\kappa]^{<\omega}$ and $f \in \Pi_{t\in F}\, \kappa_t$, then $|D^\gamma \cap U \cap H| = |D^\gamma \cap U \cap W| = \kappa$.*

Proof. (a) If $x \in Z \backslash K_\xi$ then with $t := (x, \xi)$ we have $x \in W_t^1 \in \mathcal{T}_{\mathcal{KID}}$ and $W_t^1 \cap K_\xi = \emptyset$.

(b) If $x \in K_\xi$ then with $t := (x, \xi)$ we have $W_t^1 \in \mathcal{T}_{\mathcal{K}ID}$ and $W_t^1 \cap K_\xi = \{x\}$.

(c) Let $I := \bigcap_{t\in F} I_t^{f(t)}$. Since $\mathcal{I}$ is $\kappa$-independent we have $|I| = \kappa$. For each $\eta \in I$ the set $D^\gamma \cap H$ contains the set $D_\eta^\gamma$; since the sets $D_\eta^\gamma$ $(\eta \in I)$ are pairwise disjoint, each dense in $(X, \mathcal{T})$, we have

$$\kappa = |X| \geq |D^\gamma \cap U \cap H| \geq |I| = \kappa. \qquad (*)$$

It remains to show that $|D^\gamma \cap U \cap W| = \kappa$. First, set

$$K := \bigcup_{(x,\xi)\in F} K_\xi \text{ and } L := \bigcup_{(x,\xi)\in F} (K_\xi \cup \{x\}),$$

and note from (4) and (5) of 2.4 that $\mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{ID}})}(D^\gamma \cap K) = \emptyset$, hence also

$$\mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{ID}})}(D^\gamma \cap L) = \emptyset \qquad (**)$$

(since $(D^\gamma, \mathcal{T}^{\mathcal{ID}})$ is crowded).

Now let $A := (D^\gamma \backslash L) \cap (U \cap H)$. Since $D^\gamma \cap U \cap W \supseteq A$, it suffices to show $|A| = \kappa$. If $A \in [X]^{<\kappa}$ then, writing $S := \{\eta < \kappa : A \cap D_\eta^\gamma \neq \emptyset\}$, we have $|S| \leq |A| < \kappa$, so by 2.4(3) there is $\widetilde{t} \in (Z \times 2^\kappa) \backslash F$ such that $S \subseteq I_{\widetilde{t}}^0$; then $S \cap I_{\widetilde{t}}^1 = \emptyset$ and hence $A \cap H_{\widetilde{t}}^1 = \emptyset$. Then with

$$\widetilde{f} := f \cup \{(\widetilde{t}, 1)\} \in \Pi_{t\in F\cup\{\widetilde{t}\}}\, \kappa_t \text{ and}$$
$$\widetilde{H} := \bigcap_{t\in F\cup\{\widetilde{t}\}} H_t^{f(t)} = H \cap H_{\widetilde{t}}^1 \in \mathcal{H}$$

we have $\emptyset = A \cap H_{\widetilde{t}}^1 = (D^\gamma \backslash L) \cap (U \cap H) \cap H_{\widetilde{t}}^1 = (D^\gamma \backslash L) \cap (U \cap \widetilde{H})$ and hence

$$\begin{aligned} D^\gamma \cap L &\supseteq (D^\gamma \cap L) \cap (U \cap \widetilde{H}) = \emptyset \cup [(D^\gamma \cap L) \cap (U \cap \widetilde{H})] \\ &= [(D^\gamma \backslash L) \cap (U \cap \widetilde{H})] \cup [(D^\gamma \cap L) \cap (U \cap \widetilde{H})] \\ &= D^\gamma \cap U \cap \widetilde{H}. \end{aligned} \qquad (***)$$

By (*) applied with $\widetilde{H}$ replacing $H$, the set $D^\gamma \cap U \cap \widetilde{H}$ is a nonempty $\mathcal{T}^{\mathcal{ID}}$-open subset of $D^\gamma$, so (***) contradicts (**). □

**Corollary 2.6** [With the conventions of 2.2 and 2.4.]

(a) *$(D^\gamma, \mathcal{T}^{\mathcal{ID}})$ is crowded, and $D^\gamma$ is dense in $(X, \mathcal{T}^{\mathcal{ID}})$;*

(b) *$(D^\gamma, \mathcal{T}_{\mathcal{KID}})$ is crowded, and $D^\gamma$ is dense in $(X, \mathcal{T}_{\mathcal{KID}})$; and*

(c) *$\Delta(X, \mathcal{T}^{\mathcal{ID}}) = \Delta(X, \mathcal{T}_{\mathcal{K}ID}) = \Delta(X, \mathcal{T}) = \kappa$.*

Proof. The inequalities $\Delta(X, \mathcal{T}^{\mathcal{ID}}) \leq \Delta(X, \mathcal{T}) = \kappa$ and $\Delta(X, \mathcal{T}_{\mathcal{KID}}) \leq \Delta(X, \mathcal{T}) = \kappa$ of (c) follow from the inclusions $\mathcal{T} \subseteq \mathcal{T}^{\mathcal{ID}}$ and $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{KID}}$, and all else is immediate from Lemma 2.5. □

It is easily seen that each infinite (Hausdorff) space $(X, \mathcal{T})$ contains an infinite cellular family, hence satisfies $S(X, \mathcal{T}) \geq \omega^+$. According to a result of Erdős and Tarski [17] (see also [11](3.5), [12](2.14)) every infinite Souslin number is regular. That allows us to compute exactly numbers of the form $S(X, \mathcal{T}_{\mathcal{KID}})$ in terms of the number $S(X, \mathcal{T})$ and the family $\{\kappa_t : t \in Z \times 2^\kappa\}$.

**Lemma 2.7** [With the conventions of 2.2 and 2.4.]
*$S(X, \mathcal{T}_{\mathcal{KID}})$ is the smallest regular cardinal $\kappa'$ such that*
(i) $\kappa' \geq S(X, \mathcal{T})$, *and*
(ii) $t \in Z \times 2^\kappa \Rightarrow \kappa' \geq \kappa_t^+$.

Proof. From $\mathcal{T} \subseteq \mathcal{T}_{\mathcal{KID}}$ follows $S(X, \mathcal{T}) \leq S(X, \mathcal{T}_{\mathcal{KID}})$. Further for $t \in Z \times 2^\kappa$ the family $\{W_t^\alpha : \alpha < \kappa_t\}$ is cellular in $(X, \mathcal{T}_{\mathcal{KID}})$, so $S(X, \mathcal{T}_{\mathcal{KID}}) \geq \kappa_t^+$. Since $S(X, \mathcal{T}_{KID})$ is regular by the cited theorem of Erdős and Tarski, we have $S(X, \mathcal{T}_{\mathcal{KID}}) \geq \kappa'$.

Suppose now that $\{U_\zeta \cap W_\zeta : \zeta < \kappa'\}$ is a faithfully indexed cellular family of $\mathcal{T}_{\mathcal{KID}}$-basic open subsets of $X$; here $U_\zeta \in \mathcal{T}$ and $W_\zeta = \bigcap_{t \in F_\zeta} W_t^{f_\zeta(t)}$ with $F_\zeta \in [Z \times 2^\kappa]^{<\omega}$, $f_\zeta \in \Pi_{t \in F_\zeta} \kappa_t$, $W_t^{f_\zeta(t)} \in \mathcal{W}$. Since $\{F_\zeta : \zeta < \kappa'\}$ is a family of finite sets indexed (not necessarily faithfully) by the regular cardinal $\kappa'$, there are $A \in [\kappa']^{\kappa'}$ and a set $F$ such that $F_{\zeta_0} \cap F_{\zeta_1} = F$ for every pair $\{\zeta_0, \zeta_1\} \in [A]^2$. (See [11] or [12] or [24] for proofs and bibliographic commentary on this theorem, its special cases and generalizations.) Since $|F| < \omega$ and $f_\zeta(t) < \kappa_t < \kappa'$ for each $\zeta \in A$ and $t \in F$, there is $B \in [A]^{\kappa'}$ such that $f_{\zeta_0}(t) = f_{\zeta_1}(t)$ for all $\zeta_0, \zeta_1 \in B$ and $t \in F$. We define
$f : F_{\zeta_0} \cup F_{\zeta_1} \to \bigcup_{t \in F_{\zeta_0} \cup F_{\zeta_1}} \kappa_t$
by

$$f(t) = \left\{ \begin{array}{rcl} f_{\zeta_0}(t) = f_{\zeta_1}(t) & \text{if} & t \in F \\ f_{\zeta_0}(t) & \text{if} & t \in F_{\zeta_0} \backslash F \\ f_{\zeta_1}(t) & \text{if} & t \in F_{\zeta_1} \backslash F \end{array} \right\}.$$

(More succintly: $f = f_{\zeta_0} | F_{\zeta_0} \cup f_{\zeta_1} | F_{\zeta_1}$.) Then since $S(X, \mathcal{T}) \leq \kappa' = |B|$ there are distinct $\zeta_0, \zeta_1$ (henceforth fixed) in $B$ such that $U_{\zeta_0} \cap U_{\zeta_1} \neq \emptyset$.

Then $H_{\zeta_0} \cap H_{\zeta_1} = \bigcap_{t \in F_0 \cup F_1} H_t^{f(t)}$, and (using (c) in Lemma 2.5) we have
$\emptyset \neq (H_{\zeta_0} \cap H_{\zeta_1}) \cap (U_{\zeta_0} \cap U_{\zeta_1}) \in \mathcal{T}^{\mathcal{ID}}$.
Now choose and fix $\gamma < \tau$, and (arguing much as in the proof of Lemma 2.5(c)) set
$K := \bigcup_{(x,\xi) \in F_{\zeta_0} \cup F_{\zeta_1}} K_\xi$ and $L := \bigcup_{(x,\xi) \in F_{\zeta_0} \cup F_{\zeta_1}} (K_\xi \cup \{x\})$;
then $K \in \mathcal{K}$ by 2.4(4) and $D^\gamma \backslash K$ is dense in the crowded space $(D^\gamma, \mathcal{T}^{\mathcal{ID}})$ by 2.4(5), so $D^\gamma \backslash L$ is also dense in $(D^\gamma, \mathcal{T}^{\mathcal{ID}})$, hence also in $(X, \mathcal{T}^{\mathcal{ID}})$ by Corollary 2.6(a). Then
$(D^\gamma \backslash L) \cap (H_{\zeta_0} \cap H_{\zeta_1}) \cap (U_{\zeta_0} \cap U_{\zeta_1}) \neq \emptyset$,
so
$(D^\gamma \backslash L) \cap (W_{\zeta_0} \cap W_{\zeta_1}) \cap (U_{\zeta_0} \cap U_{\zeta_1}) \neq \emptyset$,
contrary to the condition $(W_{\zeta_0} \cap U_{\zeta_0}) \cap (W_{\zeta_1} \cap U_{\zeta_1}) = \emptyset$. □

**Discussion 2.8** The method of $\mathcal{KID}$ expansion was introduced in [21] and was used in [22] to give the existence, assuming Lusin's Hypothesis, of $\omega$-resolvable Tychonoff spaces which are not maximally resolvable. The present authors have used the method subsequently [7], [8] to find and construct explicit spaces with some of the properties given in the Abstract. Arguments with some similar features were found independently and exploited by Juhász, Szentmiklossy, and Soukup [25]; we note that [25] was submitted to the journal of record before [8] was submitted, furthermore the date of publication of [25] precedes that of [8].

The principal thrust of the present paper is this: Not only do specific spaces (constructed as in [21], [22], [7], [25], [8]) exist with the properties listed, but indeed every crowded Tychonoff space subject to minimal necessary conditions admits such Tychonoff expansions.

**Definition 2.9** [With the conventions of 2.2, but with $\mathcal{K}$ not yet defined.] Let $\mathcal{M} = \{M_\xi : \xi < 2^\kappa\} \subseteq \mathcal{P}(Z)$ with $M_0 = \emptyset$. Then $\widetilde{\mathcal{M}} = \{\widetilde{M_\xi} : \xi < 2^\kappa\}$ is defined as follows.

$\widetilde{M_0} = \emptyset$, and

if $0 < \xi < 2^\kappa$ and $\widetilde{M_\eta}$ has been defined for all $\eta < \xi$ then

$\widetilde{M_\xi} = M_\xi$ if each set of the form

$(M_\xi \cup \widetilde{M_{\eta_0}} \cup \widetilde{M_{\eta_1}} \cup \cdots \cup \widetilde{M_{\eta_m}}) \cap D^\gamma$ $(m < \omega,\ \eta_i < \xi,\ \gamma < \tau)$

has empty interior in the space $(D^\gamma, \mathcal{T}^{\mathcal{ID}})$,

$\widetilde{M_\xi} = \emptyset$ otherwise.

**Lemma 2.10** *Let $Y$ be a crowded (Hausdorff) space and let $E = \bigcup_{i\leq m} E_i \subseteq Y$ with each $E_i$ discrete, $m < \omega$. Then $\mathrm{int}_Y\, E = \emptyset$.*

Proof. This is clear when $m = 0$. Suppose it holds for $m = k$ and let $E = \bigcup_{i\leq k+1} E_i \subseteq Y$ with each $E_i$ discrete. Suppose for a contradiction that there is $p \in \mathrm{int}_Y\, E$, say with $p \in E_{k+1}$, and find open $U \subseteq Y$ such that $U \cap E_{k+1} = \{p\}$. Then $(U \cap \mathrm{int}_Y\, E) \cap E_{k+1} = \{p\}$, so $\bigcup_{i\leq k} E_i$ contains the nonempty open set $(U \cap \mathrm{int}_Y\, E)\backslash\{p\}$. $\square$

**Theorem 2.11** [With the conventions of 2.2 and 2.4(1), (2), (3).]
*Let $\mathcal{M} = \{M_\xi : \xi < 2^\kappa\} = \mathcal{P}(Z)$ and let $\mathcal{K} := \widetilde{\mathcal{M}} = \{\widetilde{M_\xi} : \xi < 2^\kappa\}$. Then*

(a) *$\mathcal{K}$ satisfies conditions* (4) *and* (5) *of* 2.4*;*

(b) *if $\overline{\xi} < 2^\kappa$ and $\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(M_{\overline{\xi}} \cap D^\gamma) = \emptyset$ for all $\gamma < \tau$, then $M_{\overline{\xi}} = \widetilde{M_{\overline{\xi}}} \in \mathcal{K}$; and*

(c) *each space $(D^\gamma \cap Z, \mathcal{T}_{\mathcal{KID}})$ is hereditarily irresolvable.*

Proof. (a) is obvious.

(b) Fix $\overline{\xi} < 2^\kappa$ and $\gamma < \tau$, and let $\eta_0, \eta_1, \cdots \eta_m < \overline{\xi}$, $\emptyset \neq U \in \mathcal{T}$ and $H = \bigcap_{t\in F} H_t^{f(t)}$ with $F \in [Z \times 2^\kappa]^{<\omega}$, $f \in \Pi_{t\in F}\, \kappa_t$. We must show that if $\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(M_{\overline{\xi}} \cap D^\gamma) = \emptyset$ for all $\gamma < \tau$, then

$$\mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{ID}})}((M_{\overline{\xi}} \cup \widetilde{M_{\eta_0}} \cup \widetilde{M_{\eta_1}} \cup \ldots \cup \widetilde{M_{\eta_m}}) \cap D^\gamma) = \emptyset. \qquad (*)$$

Writing $W = \bigcap_{t\in F} W_t^{f(t)}$, we have, since $D^\gamma \backslash M_{\overline{\xi}}$ is dense in $(D^\gamma, \mathcal{T}_{\mathcal{KID}})$ and $\emptyset \neq U \cap W \in \mathcal{T}_{\mathcal{KID}}$, that

$Y := (D^\gamma \backslash M_{\overline{\xi}}) \cap (U \cap W)$ is dense in $((D^\gamma \cap (U \cap W)), \mathcal{T}_{\mathcal{KID}})$.

Further since $(D^\gamma \cap (U \cap W), \mathcal{T}_{\mathcal{KID}})$ is crowded, its dense subset $(Y, \mathcal{T}_{\mathcal{KID}})$ is crowded.

We have $W\backslash H \subseteq L := \bigcup_{(x,\xi)\in F} (K_\xi \cup \{x\})$, with $L$ the union of finitely many discrete subsets of $(Z, \mathcal{T}_{\mathcal{KID}}) \subseteq (X, \mathcal{T}_{\mathcal{KID}})$. Each $\widetilde{M_{\eta_i}} \in \mathcal{K}$ is also discrete in $(Z, \mathcal{T}_{\mathcal{KID}}) \subseteq (X, \mathcal{T}_{\mathcal{KID}})$, so from Lemma 2.10 it follows that the set $Y\backslash(\bigcup_{i\leq m} \widetilde{M_{\eta_i}} \cup L)$ remains dense in $(D^\gamma \cap U \cap W, \mathcal{T}_{\mathcal{KID}})$, and (*) follows.

(c) Suppose for some $\gamma_0 < \tau$ there are $\xi_0 < 2^\kappa$ and nonempty $S \subseteq D^{\gamma_0} \cap Z$ such that $M_{\xi_0} \subseteq S$ and both $M_{\xi_0}$ and $S\backslash M_{\xi_0}$ are dense in $(S, \mathcal{T}_{\mathcal{KID}})$. From $\mathrm{int}_{(S,\mathcal{T}_{\mathcal{KID}})} M_{\xi_0} = \emptyset$

it follows that $\mathrm{int}_{(D^{\gamma_0},\mathcal{T}_{\mathcal{KID}})}\widetilde{M_{\xi_0}} = \emptyset$, so $\mathrm{int}_{(D^{\gamma},\mathcal{T}_{\mathcal{KID}})}(M_{\xi_0} \cap D^{\gamma}) = \emptyset$ for each $\gamma < \tau$. From (b) we then have $M_{\xi_0} = \widetilde{M_{\xi_0}} \in \mathcal{K}$, so by Lemma 2.5(a) the set $M_{\xi_0}$ is closed in $(Z, \mathcal{T}_{\mathcal{KID}})$ (hence in $(S, \mathcal{T}_{\mathcal{KID}})$); this contradicts the density in $(S, \mathcal{T}_{\mathcal{KID}})$ of both $M_{\xi_0}$ and $S \backslash M_{\xi_0}$. □

# 3 The $\mathcal{KID}$ Expansion: Applications

We begin this Section by proving (the case $|X| = \Delta(X)$ of) our principal theorem (cf. item $(i = 1)$ of the Abstract). The result is in the tradition of the several papers listed in the Bibliography which respond to the Ceder-Pearson question (Is there an $\omega$-resolvable space which is not maximally resolvable?), but this has a different flavor: Not only can examples of such spaces be constructed by *ad hoc* techniques, but indeed *every* (suitably restricted) $\omega$-resolvable Tychonoff space admits a Tychonoff expansion $\mathcal{U}$ such that $(X, \mathcal{U})$ remains $\omega$-resolvable but is not maximally resolvable. For remarks intended to justify or to explain the special hypothesis "$S(X, \mathcal{T}) \leq |X|$" in Theorem 3.1, see Remark 5.3 below, where it is noted that in some settings where $S(X, \mathcal{T}) \leq |X|$ fails, $\omega$-resolvability implies maximal resolvability.

**Theorem 3.1** *Let $X = (X, \mathcal{T})$ be a crowded, $\omega$-resolvable Tychonoff space with $S(X, \mathcal{T}) \leq |X| = \Delta(X, \mathcal{T}) = \kappa$. Then there is a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) *$S(X, \mathcal{U}) = S(X, \mathcal{T})$ and $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T})$;*
(b) *$(X, \mathcal{U})$ is $\omega$-resolvable;*
(c) *$(X, \mathcal{U})$ is not maximally resolvable; and*
(d) *$(X, \mathcal{U})$ is not $S(X, \mathcal{T})$-resolvable, if $(X, \mathcal{T})$ is maximally resolvable.*

Proof. If $(X, \mathcal{T})$ is not maximally resolvable the conditions are satisfied with $\mathcal{U} := \mathcal{T}$, so we assume in what follows that $(X, \mathcal{T})$ is maximally resolvable.

Let $\mathcal{D} = \{D^n_\eta : \eta < \kappa, n < \omega\}$ be a faithfully indexed dense partition of $(X, \mathcal{T})$, and set $D^n := \bigcup_{\eta < \kappa} D^n_\eta$ for $n < \omega$. Take $Z = X$ in Definition 2.2 and let $\mathcal{I} = \{\mathcal{I}_t : t \in X \times 2^\kappa\}$ be a $\kappa$-independent family of partitions $\mathcal{I}_t$ of $X$ with the strong small-set-separating property given by Lemma 1.6; for simplicity we take $\kappa_t = 2 = \{0, 1\}$ for each $t \in X \times 2^\kappa$.

Let $\mathcal{M} = \{M_\xi : \xi < 2^\kappa\} = \mathcal{P}(X)$, and define $\mathcal{K} := \widetilde{\mathcal{M}}$ as in Definition 2.9. We will show that $\mathcal{U} := \mathcal{T}_{\mathcal{KID}}$ is as required.

(a) The equality $\Delta(X, \mathcal{T}_{\mathcal{KID}}) = \Delta(X, \mathcal{T})$ is given by Corollary 2.6, while $S(X, \mathcal{T}_{\mathcal{KID}}) = S(X, \mathcal{T})$ is immediate from Lemma 2.7 (using the regularity of $S(X, \mathcal{T})$ and the fact that $\kappa_t < \omega < \omega^+ \leq S(X, \mathcal{T})$ for each $t \in Z \times 2^\kappa$).

(b) According to Corollary 2.6(b), the disjoint sets $D^n$ $(n < \omega)$ are dense in $(X, \mathcal{T}_{\mathcal{KID}})$.

(c) and (d) Suppose there is a family $\mathcal{E}$ of pairwise disjoint dense subsets of $(X, \mathcal{T}_{\mathcal{KID}})$ such that $|\mathcal{E}| = S(X, \mathcal{T})$. Note then that for some $E \in \mathcal{E}$ we have

$\mathrm{int}_{(D^n,\mathcal{T}_{\mathcal{KID}})}(D^n \cap E) = \emptyset$ for each $n < \omega$. (*)

(Indeed otherwise we may argue as in [22](2.3), [7], [8](3.1(c)): choosing for each $E \in \mathcal{E}$ some $n(E) < \omega$ such that

$\mathrm{int}_{(D^{n(E)},\mathcal{T}_{\mathcal{KID}})}(D^{n(E)} \cap E) \neq \emptyset$,

we have from Lemma 2.7 and the regularity of $S(X, \mathcal{T}) = S(X, \mathcal{T}_{\mathcal{KID}})$ that some (fixed) $n < \omega$ satisfies

$\mathrm{int}_{(D^n,\mathcal{T}_{\mathcal{KID}})}(D^n \cap E) \neq \emptyset$ for $S(X,\mathcal{T}_{\mathcal{KID}})$-many $E \in \mathcal{E}$;

that gives $S(D^n,\mathcal{T}_{\mathcal{KID}}) > S(X,\mathcal{T}_{\mathcal{K}ID})$, which is impossible since $D^n$ is dense in $(X,\mathcal{T}_{\mathcal{KID}})$.)

Then choosing $E \in \mathcal{E}$ as in (*), we have from Theorem 2.11(b) that $E \in \mathcal{K}$, so $E$ is closed and discrete in the crowded space $(X,\mathcal{T}_{\mathcal{K}ID})$ by Lemma 2.5((a) and (b)). This contradicts the density of $E$ in $(X,\mathcal{T}_{\mathcal{KID}})$. $\square$

**Remark 3.2** The choice $\kappa_t < \kappa$ for all $t \in X \times 2^\kappa$ in (the proof of) Theorem 3.1 is essential. If $\kappa_t = \kappa$ is permitted for some $t$ then the refinement $\mathcal{U} = \mathcal{T}_{\mathcal{KID}}$ satisfies conditions (b) and (c), but as noted in the first paragraph of the proof of Lemma 2.7 we would now have $S(X,\mathcal{T}_{\mathcal{K}ID}) = \kappa^+ > S(X,\mathcal{T})$.

As is indicated in its proof, Theorem 3.1 is of interest only when the given space $(X,\mathcal{T})$ is maximally resolvable. So viewed, the case $\kappa' = S(X,\mathcal{T})$ of the following result (cf. item $(i = 2)$ of our Abstract) strengthens and improves Theorem 3.1.

**Theorem 3.3** *Let $X = (X,\mathcal{T})$ be a crowded, maximally resolvable Tychonoff space and let $\kappa'$ be a regular cardinal such that $S(X,\mathcal{T}) \leq \kappa' \leq |X| = \Delta(X,\mathcal{T}) = \kappa$. Then there is a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) *$S(X,\mathcal{U}) = \kappa'$ and $\Delta(X,\mathcal{U}) = \Delta(X,\mathcal{T}) = \kappa$;*

(b) *$(X,\mathcal{U})$ is $\tau$-resolvable for each $\tau < \kappa'$; and*

(c) *$(X,\mathcal{U})$ is not $\kappa'$-resolvable.*

Proof. [Being $\kappa$-resolvable, the space $(X,\mathcal{T})$ is surely $\kappa'$-resolvable, so in this case the topology $\mathcal{U}$ will of necessity be a strict refinement of $\mathcal{T}$.]

Let $\Lambda$ be the set of all cardinals $\tau$ such that $2 \leq \tau < \kappa'$, and let $\{\kappa_t : t \in T = X \times 2^\kappa\}$ list the elements of $\Lambda$ with each $\tau \in \Lambda$ appearing $2^\kappa$-many times. For $\tau \in \Lambda$ set $T(\tau) := \{t \in T : \kappa_t = \tau\}$. According to Lemma 1.6, there is a strong small-set-separating family $\kappa$-independent family $\mathcal{I} = \{\mathcal{I}_t : t \in X \times 2^\kappa\}$ of partitions of $\kappa$ which respects the partition $\{T(\tau) : \tau \in \Lambda\}$ of $T$.

We note that $\kappa' = \sup_{t \in T} \kappa_t^+$.

Let $\mathcal{D} = \{D^n_\eta : n < \omega, \eta < \kappa\}$ be a dense partition of $(X,\mathcal{T})$, and as usual set $D^n := \bigcup_{\eta<\kappa} D^n_\eta$.

Take $\mathcal{K}$ as in Theorem 2.11 and set $\mathcal{U} := \mathcal{T}_{\mathcal{KID}}$ (with $Z = X$). We show that $\mathcal{U}$ is as required.

(a) The equalities $\Delta(X,\mathcal{T}_{\mathcal{KID}}) = \Delta(X,\mathcal{T})$ and $S(X,\mathcal{T}_{\mathcal{KID}}) = \kappa'$ are given by Corollary 2.6(c) and Lemma 2.7, respectively.

(c) The argument showing that the space $(X,\mathcal{T}_{\mathcal{KID}})$ of Theorem 3.1(c) is not $S(X,\mathcal{T}_{\mathcal{K}ID})$-resolvable (i.e., is not $\kappa'$-resolvable) applies here *verbatim* to prove (c).

(b) Let $\mathcal{A} = \{A_n : n < \omega\}$ be an arbitrary countable dense partition of the space $(X,\mathcal{T}_{\mathcal{KID}})$. Fix $\tau < \kappa'$, let $t(n)$ $(n < \omega)$ be a faithfully indexed sequence from $X \times 2^\kappa$ such that $\kappa_{t(n)} = \tau$ for each $n < \omega$, and for $n < \omega$ and $\alpha < \tau$ set

$E^\alpha_n := W^\alpha_{t(n)} \backslash \bigcup_{k<n} W^\alpha_{t(k)}$.

Each set $E^\alpha_n$ is nonempty, and by Remark 2.3(b) each is $\mathcal{T}_{\mathcal{KID}}$-clopen. Now define

$E^\alpha := \bigcup_{n<\omega}(E^\alpha_n \cap A_n)$ $(\alpha < \tau)$;

we will show that $\{E^\alpha : \alpha < \tau\}$ is a dense partition of $(X,\mathcal{T}_{\mathcal{KID}})$.

Suppose there is $x \in E^\alpha \cap E^{\alpha'}$ with $\alpha, \alpha' < \tau$. Then there are $n, n' < \omega$ such that

$$x \in (E^\alpha_n \cap A_n) \cap (E^{\alpha'}_{n'} \cap A_{n'}) \subseteq A_n \cap A_{n'},$$

so $n = n'$ and from $x \in E^\alpha_n \cap E^{\alpha'}_n \subseteq W^\alpha_{t(n)} \cap W^{\alpha'}_{t(n)}$ we have $\alpha = \alpha'$, as required.

To see for (fixed) $\alpha < \tau$ that $E^\alpha$ is dense in $(X, \mathcal{T}_{\mathcal{KID}})$, let $U \cap W \in \mathcal{T}_{KID}$ with $\emptyset \neq U \in \mathcal{T}$ and with $W = \bigcap_{t\in F} W_t^{f(t)}$ with $F \in [X \times 2^\kappa]^{<\omega}$, $f \in \Pi_{t\in F}\,\kappa_t$. We assume without loss of generality, replacing $W$ by a smaller set if necessary, that some $t(n) \in F$; and further with $m := \max\{n : t(n) \in F\}$ that $n < m \Rightarrow t(n) \in F$. It suffices to show that $(U \cap W) \cap E^\alpha_n \neq \emptyset$ for some $n$, for then (from the density of $A_n$ in $(X, \mathcal{T}_{\mathcal{KID}})$ and the fact that $E^\alpha_n$ is open in $(X, \mathcal{T}_{\mathcal{KID}})$) it will follow that

$$(U \cap W) \cap E^\alpha \supseteq (U \cap W) \cap (E^\alpha_n \cap A_n) = (U \cap W \cap E^\alpha_n) \cap A_n \neq \emptyset.$$

<u>Case 1</u>. Some $n \leq m$ satisfies $f(t(n)) = \alpha$. Then, choosing minimal such $n$, we have $\emptyset \neq U \cap W \subseteq W \subseteq E^\alpha_n$, so $(U \cap W) \cap E^\alpha_n = U \cap W \neq \emptyset$.

<u>Case 2</u>. Case 1 fails. Then defining $\widetilde{f} := f \cup \{(t(m+1), \alpha)\}$ we have $W \cap W^\alpha_{t(m+1)} \subseteq E^\alpha_{t(m+1)}$, and Lemma 2.5(c) gives

$$\emptyset \neq U \cap (W \cap W^\alpha_{t(m+1)}) \cap E^\alpha_{t(m+1)} \subseteq (U \cap W) \cap (E^\alpha_{m+1}). \ \square$$

**Remarks 3.4** (a) According to Theorem 2.5 the family $\{D^n : n < \omega\}$ is a dense partition of $(X, \mathcal{T}_{\mathcal{KID}})$. We note that the construction just given parlays an arbitrary countable dense partition $\mathcal{A} = \{A_n : n < \omega\}$ of $(X, \mathcal{T}_{\mathcal{KID}})$ into a dense partition of $(X, \mathcal{T}_{\mathcal{KID}})$ of cardinality $\tau$. It is not necessary to assume that $\mathcal{A} = \{D^n : n < \omega\}$.

(b) The argument of Theorem 3.3(b) closely parallels our proof in [8](4.2) that an $\omega$-resolvable, dense subset $X$ of a space of the form $(D(\kappa))^I$ is necessarily $\kappa$-resolvable (i.e., is maximally resolvable in case $\Delta(X) = \kappa$). That theorem, surprising to the authors, helps to explain the difficulty encountered over the years by many workers attempting to answer the question of Ceder and Pearson [3]: Is every $\omega$-resolvable space maximally resolvable?

(c) It should be noted that a dense subspace of a space of the form $(D(\kappa))^I$ need not be $\omega$-resolvable. Indeed in [8](2.3) we show that for every $\kappa \geq \omega$ there is a dense set $X \subseteq (D(\kappa))^{2^\kappa}$ such that $|X| = \Delta(X) = \kappa$, no subset of $X$ is resolvable, and every dense subset of $X$ is open in $X$. See also [1](2.3), [6](5.4) and [25](4.1) for parallel results in the space $\{0,1\}^{2^\kappa}$.

(d) A propos of (b) above, we note that other criteria sufficient to ensure maximal resolvability have been established by other authors. For example, years ago Pytke′ev [31] showed that every $k$-space, also every space $X$ for which the tightness $t(X)$ satisfies $t(X) < \Delta(X)$, is maximally resolvable. More recently, denoting by $\mathrm{ps}(X)$ the smallest successor cardinal such that every discrete set $S \subseteq X$ satisfies $|S| < \mathrm{ps}(X)$, Pavlov [29] showed that every $T_1$-space such that $\Delta(X) > \mathrm{ps}(X)$ is maximally resolvable. That theorem was strengthened in two ways in [26]: No separation hypothesis on $X$ is required, and maximal resolvability of $X$ is established assuming only $\Delta(X) \geq \mathrm{ps}(X)$.

Our proof of Theorem 3.3 rests on the conventions of Section 2, and uses crucially the (strong) hypothesis that $(X, \mathcal{T})$ is maximally resolvable. That hypothesis can be weakened to the assumption that $(X, \mathcal{T})$ is $\kappa'$-resolvable, with $\kappa'$ regular and $S(X, \mathcal{T}) \leq \kappa' \leq |X| = \Delta(X, \mathcal{T}) = \kappa$, provided that the equality $2^{\kappa'} = 2^{|X|}$ is assumed. Indeed the argument given in the proof of Theorem 3.3 shows that $\mathcal{U} := \mathcal{T}_{\mathcal{KID}}$ has properties

(a), (b) and (c), with $\mathcal{D} = \{D^n_\eta : n < \omega, \eta < \kappa'\}$ a dense partition of $(X, \mathcal{T})$, with $\mathcal{I} = \{\mathcal{I}_t : t \in X \times 2^{\kappa'}\}$ a strong small-set-separating, $\kappa'$-independent family of partitions of $\kappa'$, and with $\mathcal{K} = \widetilde{\mathcal{M}}$ as in Definition 2.9 with $Z = X$. We do not know in ZFC whether the hypothesis of Theorem 3.3 can be weakened. Specifically we ask:

**Question 3.5** Let $X = (X, \mathcal{T})$ be a crowded Tychonoff space and let $\kappa'$ be a regular cardinal such that $S(X, \mathcal{T}) \leq \kappa' < |X| = \Delta(X, \mathcal{T}) = \kappa$ and $(X, \mathcal{T})$ is $\tau$-resolvable for each $\tau < \kappa'$. Must there then exist, in ZFC, a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that

(a) $S(X, \mathcal{U}) \leq \kappa'$ (perhaps even: $S(X, \mathcal{U}) = S(X, \mathcal{T})$) and $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T}) = \kappa$);

(b) $(X, \mathcal{U})$ is $\tau$-resolvable for each $\tau < \kappa'$; and

(c) $(X, \mathcal{U})$ is not $\kappa'$-resolvable?

Of course, Question 3.5 is of interest only if $(X, \mathcal{T})$ is itself $\kappa'$-resolvable, since otherwise $\mathcal{U} := \mathcal{T}$ would be as required.

Next we prove item $(i = 3)$ of the Abstract for the case $|X| = \Delta(X)$.

**Theorem 3.6** *Let $X = (X, \mathcal{T})$ be a crowded, maximally resolvable Tychonoff space with $S(X, \mathcal{T}) \leq |X| = \Delta(X, \mathcal{T}) = \kappa$. Then there is a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) $S(X, \mathcal{U}) = S(X, \mathcal{T})$ *and* $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T})$;

(b) $(X, \mathcal{U})$ *is maximally resolvable; and*

(c) $(X, \mathcal{U})$ *is not extraresolvable.*

Proof. We invoke the conventions of 2.2 and 2.4, now taking $\tau = \kappa$.

Let $\mathcal{D} = \{D^\gamma_\eta : \eta < \kappa, \gamma < \kappa\}$ be a faithfully indexed dense partition of $(X, \mathcal{T})$, and set $D^\gamma := \bigcup_{\eta<\kappa} D^\gamma_\eta$ for $\gamma < \kappa$. Let $\mathcal{I} = \{\mathcal{I}_t : t \in X \times 2^\kappa\}$ be a $\kappa$-independent family of partitions $\mathcal{I}_t$ of $X$ with the strong small-set-separating property; for simplicity we take $\kappa_t = 2 = \{0, 1\}$ for each $t \in X \times 2^\kappa$.

Let $\mathcal{M} = \{M_\xi : \xi < 2^\kappa\} = \mathcal{P}(X)$, and define $\mathcal{K} := \widetilde{\mathcal{M}}$ as in Definition 2.9 (taking $Z = X$). We will show that $\mathcal{U} := \mathcal{T}_{\mathcal{KID}}$ is as required.

(a) The equality $\Delta(X, \mathcal{T}_{\mathcal{KID}}) = \Delta(X, \mathcal{T})$ is given by Corollary 2.6, while $S(X, \mathcal{T}_{\mathcal{KID}}) = S(X, \mathcal{T})$ is immediate from Lemma 2.7 (using the regularity of $S(X, \mathcal{T})$ and the fact that $\kappa_t < \omega < \omega^+ \leq S(X, \mathcal{T})$ for each $t \in Z \times 2^\kappa$).

(b) According to Corollary 2.6(b), the disjoint sets $D^\gamma$ $(\gamma < \kappa)$ are dense in $(X, \mathcal{T}_{\mathcal{KID}})$.

(c) Suppose there is a family $\mathcal{E}$ of dense subsets of $(X, \mathcal{T}_{\mathcal{KID}})$, with $|\mathcal{E}| = \kappa^+$, such that every two elements of $\mathcal{E}$ have intersection which is nowhere dense in $(X, \mathcal{T}_{\mathcal{KID}})$. We claim that, much as in the proof of Theorem 3.1(c), there is $E \in \mathcal{E}$ such that

$$\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E) = \emptyset \text{ for each } \gamma < \kappa. \qquad (*)$$

For if (*) fails then some (fixed) $\gamma < \kappa$ satisfies

$$\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E) \neq \emptyset \text{ for } \kappa^+\text{-many } E \in \mathcal{E},$$

and then since $S(D^\gamma, \mathcal{T}_{\mathcal{KID}}) = S(X, \mathcal{T}_{\mathcal{K}ID}) = S(X, \mathcal{T}) \leq \kappa$ there are distinct $E, E' \in \mathcal{E}$ such that

$$\emptyset \neq [\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E)] \cap [\mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E')] = \mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E \cap E').$$

Then with $\mathcal{T}_{\mathcal{KID}}$-open $U \subseteq X$ chosen so that $D^\gamma \cap U = \mathrm{int}_{(D^\gamma, \mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E \cap E')$ we have

$\emptyset \neq U \subseteq \mathrm{cl}_{(X,\mathcal{T}_{\mathcal{KID}})}U = \mathrm{cl}_{(X,\mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap U) = \mathrm{cl}_{(X,\mathcal{T}_{\mathcal{KID}})}\mathrm{int}_{(X,\mathcal{T}_{\mathcal{KID}})}(D^\gamma \cap E \cap E') \subseteq \mathrm{cl}_{(X,\mathcal{T}_{\mathcal{KID}})}(E \cap E')$,
contrary to the fact that $E \cap E'$ is nowhere dense in $(X, \mathcal{T}_{\mathcal{KID}})$. Thus (*) is established.

Then, choosing $E \in \mathcal{E}$ as in (*), we have from Theorem 2.11(b) (applied to the set $M_{\overline{\xi}} = E$) that $E \in \mathcal{K} = \widetilde{\mathcal{M}}$, so by Lemma 2.5((a) and (b)) the set $E$ is closed and discrete in the crowded space $(X, \mathcal{T}_{\mathcal{K}ID})$. This contradicts the density of $E$ in $(X, \mathcal{T}_{\mathcal{KID}})$. $\square$

We turn next to establishing items ($i = 4$) and ($i = 5$) of the Abstract for the case $|X| = \Delta(X)$. As expected, refinements of the form $\mathcal{U} = \mathcal{T}_{\mathcal{KID}}$ play a central role; it is necessary only to tailor in each case the specifics of the families $\mathcal{K}$, $\mathcal{I}$, and $\mathcal{D}$ to the task at hand. But in Theorem 3.10 the process is iterated: a first expansion $\mathcal{T}' \supseteq \mathcal{T}$ satisfies $\mathrm{nwd}(X, \mathcal{T}') = \kappa$, a second expansion $\mathcal{T}'' \supseteq \mathcal{T}'$ is maximally resolvable but not extraresolvable, and a final expansion (of the form $\mathcal{T}''^{\mathcal{ID}}$, not $\mathcal{T}''_{\mathcal{KID}}$) has all required properties.

For the proofs of (the case $|X| = \Delta(X)$ of) items ($i = 4$) and ($i = 5$) of the Abstract, we need two preliminary lemmas. A weak version of Lemma 3.7 is proved in our work [7](3.9). A strictly combinatorial proof exists, but it is lengthy; we give instead an argument which uses the topological constructions already at our disposal.

**Lemma 3.7** *Let $\tau \geq \omega$. There exist families $\mathcal{A} = \{\mathcal{A}_\xi : \xi < 2^\tau\}$ and $\mathcal{S}_{er} \subseteq \mathcal{P}(\tau)$ such that*

(i) *$\mathcal{A}$ is a $\tau$-independent family of partitions of $\tau$ with the strong small-set-separating property, with each $\mathcal{A}_\xi \in \mathcal{A}$ of the form $\mathcal{A}_\xi = \{A^0_\xi, A^1_\xi\}$;*

(ii) $|\mathcal{S}_{er}| = 2^\tau$;

(iii) *if $n < \omega$ and $S, S_1, S_2, \ldots S_n$ are distinct elements of $\mathcal{S}_{er}$ and $A = \bigcap_{\xi \in F} A^{f(\xi)}_\xi$ with $F \in [2^\tau]^{<\omega}$ and $f \in \{0,1\}^F$, then $|A \cap (S \backslash (S_1 \cup S_2 \cup \ldots \cup S_n))| = \tau$; and*

(iv) *if $S, S' \in \mathcal{S}_{er}$ with $S \neq S'$ then* (a) *for each $x \in \tau \backslash (S \cap S')$ there are infinitely many $\xi < 2^\tau$ such that $x \in A^1_\xi$ and $S \cap S' \subseteq A^0_\xi$; and* (b) *for each $x \in S \cap S'$ there are infinitely many $\xi < 2^\tau$ such that $(S \cap S') \cap A^1_\xi = \{x\}$.*

Proof. Let $\mathcal{J} \cup \mathcal{L} \cup \{\mathcal{D}\}$ be a $\tau$-independent family of partitions of $\tau$, where $\mathcal{J} = \{\mathcal{J}_\xi : \xi < 2^\tau\}$ is chosen (as in Theorem 1.9) so that the space

$Y = (Y, \mathcal{T}) := e_{\mathcal{J}}[\tau] \subseteq K := \{0,1\}^{\mathcal{J}} = \{0,1\}^{2^\tau}$

has properties (a), (b), (c) and (d) of Theorem 1.9. We take $|\mathcal{J}| = |\mathcal{L}| = 2^\tau$, say $\mathcal{J} = \{\mathcal{J}_\xi : \xi < 2^\tau\}$ and $\mathcal{L} = \{\mathcal{L}_\zeta : \zeta < 2^\tau\}$, and we take each $\mathcal{J}_\xi \in \mathcal{J}$ of the form $\mathcal{J}_\xi = \{J^0_\xi, J^1_\xi\}$ and each $\mathcal{L}_\zeta \in \mathcal{L}$ of the form $\mathcal{L}_\zeta = \{L^0_\zeta, L^1_\zeta\}$.

We write $\mathcal{D} = \{D^\gamma_\eta : \gamma < \tau, \eta < \tau\}$.

The families $\mathcal{A}$ and $\mathcal{S}_{er}$ will be defined with the help of a suitable expansion $\mathcal{T}_{\mathcal{K}ID}$ of $\mathcal{T}$.

The family $\mathcal{D}$ has already been defined, and for $\mathcal{I}$ we choose an arbitrary $\tau$-independent family $\mathcal{I} = \{\mathcal{I}_t : t \in Y \times 2^\tau\}$ of partitions of $\tau$ with the strong small-set-separating property, say with each $\mathcal{I}_t$ of the form $\mathcal{I}_t = \{I^0_t, I^1_t\}$. For $\mathcal{K}$, first let

$\mathcal{K}' := \{\bigcap_{\zeta \in F} L^0_\zeta : |F| > 1, F \in [2^\tau]^{<\omega}\}$

and let $\mathcal{K}$ be the set of sets of the form $\bigcup_{i<n} K'_i$ with $n < \omega$, $K'_i \in \mathcal{K}'$. We write $\mathcal{K} = \{K_\xi : \xi < 2^\tau\}$, the indexing chosen so that each $K \in \mathcal{K}$ is listed infinitely often.

With these definitions, writing as usual $D^\gamma := \bigcup_{\eta<\tau} D^\gamma_\eta$ for $\gamma < \tau$, conditions (1), (2), (3) and (4) of 2.4 are clearly satisfied. To verify (5), fix $K \in \mathcal{K}$ and $\gamma < \tau$; we show that $\mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{I}\mathcal{D}})}(K \cap D^\gamma) = \emptyset$.

There are $n < \omega$ and $F_i \in [2^\tau]^{<\omega}$ with $|F_i| > 1$ such that $K = \bigcup_{i<n}(\bigcap_{\zeta \in F_i} L^0_\zeta)$. With $F := \bigcup_{i<n} F_i$ and $B := \bigcap_{\zeta\in F} L^1_\zeta$ we have (since $\mathcal{L} \cup \{\mathcal{D}\}$ is $\tau$-independent) for each $\eta < \tau$ that $|B \cap D^\gamma_\eta| = \tau$, so $B \cap D^\gamma$ meets each set of the form $H^\alpha_t = X(I^\alpha_t)$ with $I^\alpha_t \in \mathcal{I}_t \in \mathcal{I}$. Thus $B \cap D^\gamma$ is dense in $(D^\gamma, \mathcal{T}^{\mathcal{I}\mathcal{D}})$, and from $B \cap K = \emptyset$ it then follows that $\mathrm{int}_{(D^\gamma, \mathcal{T}^{\mathcal{I}\mathcal{D}})}(K \cap D^\gamma) = \emptyset$. Thus (5) is proved.

Now with $\mathcal{W} = \{\mathcal{W}_t : t \in Y \times 2^\tau\}$ defined (using $\mathcal{I}$ and $\mathcal{K}$) as in Definition 2.2 we set $\mathcal{A} := \mathcal{J} \cup \mathcal{W}$, and $\mathcal{S}_{er} := \{L^0_\zeta : \zeta < 2^\tau\}$. It is clear for distinct $S, S' \in \mathcal{S}_{er}$ that $S \cap S' \in \mathcal{K}' \subseteq \mathcal{K}$.

Each $\mathcal{J}_\xi \in \mathcal{J}$, and each $\mathcal{W}_t \in \mathcal{W}$, is a partition of $\tau$. Since $\mathcal{J}$ has the strong small-set-separating property, and $\mathcal{J} \subseteq \mathcal{A}$, also $\mathcal{A}$ has the strong small-set-separating property. Thus to prove (iii) and to complete the proof of (i) it suffices to show: For each triple $(J, W, L)$, with

$J = \bigcap_{\xi\in F_0} J^{f_0(\xi)}_\xi$ with $F_0 \in [2^\tau]^{<\omega}$, $f_0 \in \{0,1\}^{F_0}$,

$W = \bigcap_{t\in F_1} W^{f_1(t)}_t$ with $F_1 \in [Y \times 2^\tau]^{<\omega}$, $f_1 \in \{0,1\}^{F_1}$, and

$L = L^0_{\overline{\zeta}} \backslash \bigcup_{i<n} L^0_{\zeta_i} = L^0_{\overline{\zeta}} \cap \bigcap_{i<n} L^1_{\zeta_i}$ with distinct $\overline{\zeta}, \zeta_i < 2^\tau$,

that $|J \cap W \cap L| = \tau$.

To do that, take $|F_1| = m$, say $F_1 = \{t_j = (x_j, \xi_j) : j < m\}$, and note with $K_{\xi_j} = \bigcup_{i<n_j}(\bigcap_{\zeta\in F_{i,j}} L^0_\zeta)$ that $L \backslash K_{\xi_j}$ contains the set $C := L \cap \bigcap\{L^1_\zeta : \zeta \in \bigcup_{i<n_j} F_{i,j}\}$. Thus $L \cap W \supseteq C \cap W = C \cap H$, where $H = \bigcap_{t\in F_1} H^{f_1(t)}_t$. Since $\mathcal{J} \cup \mathcal{L} \cup \{\mathcal{D}\}$ is $\tau$-independent, and each $H \in \mathcal{H}_t \subseteq \mathcal{T}^{\mathcal{I}\mathcal{D}}$ is the union of sets in $\mathcal{D}$, the family $\mathcal{J} \cup \mathcal{H} \cup \mathcal{L}$ is also $\tau$-independent. Now $J$ is a Boolean combination of sets from $\mathcal{J}$, $H$ is a Boolean combination of sets from $\mathcal{H}$, and $C$ is a Boolean combination of sets from $\mathcal{L}$, so from $J \cap W \cap L \supseteq J \cap W \cap C = J \cap H \cap C$ then follows $|J \cap W \cap L| = \tau$, as required.

For (iv), let $S, S' \in \mathcal{S}_{er}$ with $S \neq S'$ and fix $x \in Y$. Then for each of the (infintely many) $\xi < 2^\tau$ such that $S \cap S' = K_\xi$ we have, taking $t = (x, \xi)$: If $x \notin S \cap S'$ then $x \in W^1_t$ and $S \cap S' \subseteq W^0_t$ with $W^0_t, W^1_t \in \mathcal{W}_t \subseteq \mathcal{A}$, while if $x \in S \cap S'$ then $(S \cap S') \cap W^1_t = \{x\}$ with $W^1_t \in \mathcal{W}_t \subseteq \mathcal{A}$. Then to achieve (iv) in the form stated, it is enough to re-index $\mathcal{A}$ in the form $\mathcal{A} = \{\mathcal{A}_\xi : \xi < 2^\tau\}$ $\square$

**Theorem 3.8** *Let $\tau \leq \kappa$ and let $(X, \mathcal{T})$ be a crowded, $\tau$-resolvable Tychonoff space such that $S(X, \mathcal{T}) \leq |X| = \Delta(X, \mathcal{T}) = \kappa$. Then there is a Tychonoff expansion $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) *$S(X, \mathcal{U}) = S(X, \mathcal{T})$ and $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T})$;*

(b) *$(X, \mathcal{U})$ is $\tau$-resolvable; and*

(c) *$(X, \mathcal{U})$ is $2^\tau$-extraresolvable.*

Proof. Let $\mathcal{A}$ and $\mathcal{S}_{er}$ be families given by Lemma 3.7. Ignoring the indexing there, we choose $Z \subseteq X$ with $|Z| = 1$ and we write $\mathcal{A} = \{\mathcal{A}_t : t \in Z \times 2^\tau\}$, with each $\mathcal{A}_t = \{A^0_t, A^1_t\}$. Let $\mathcal{K} = \{K_\xi : \xi < 2^\tau\}$ with each $K_\xi = \emptyset$, and let $\mathcal{D} := \{D^\gamma_\eta : \eta < \tau, \gamma < \tau\}$ be a partition of $X$ witnessing the $\tau$-resolvability of $(X, \mathcal{T})$. We show that the expansion $\mathcal{U} := \mathcal{T}^{\mathcal{A}\mathcal{D}}$ satisfies conditions (a), (b), and (c).

(a) That $\Delta(X,\mathcal{U}) = \Delta(X,\mathcal{T})$ is given by Corollary 2.6(c), while $S(X,\mathcal{U}) = S(X,\mathcal{T})$ is immediate from Lemma 2.7 (using the regularity of $S(X,\mathcal{T})$ and the fact that $\kappa_t = 2 < \omega < \omega^+ \leq S(X,\mathcal{T})$ for each $t \in Z \times 2^\tau$). Thus (a) holds.

(b) As usual, Lemma 2.5(c) shows that $\{D^\gamma : \gamma < \tau\}$ is a dense partition of $(X,\mathcal{U})$.

(c) It suffices to show that

(i) if $S \in \mathcal{S}_{er}$ then $X(S)$ is dense in $(X,\mathcal{U})$; and

(ii) if $S, S'$ are distinct elements of $\mathcal{S}_{er}$ then $X(S \cap S')$ is closed and nowhere dense in $(X,\mathcal{U})$.

For (i), given $\emptyset \neq U \in \mathcal{T}$ and $H = \bigcap_{t\in F} H_t^{f(t)}$ with $F \in [Z \times 2^\tau]^{<\omega}$ and $f \in \{0,1\}^F$, we must show $X(S) \cap (U \cap H) \neq \emptyset$. Set $A = \bigcap_{t\in F} A_t^{f(t)}$, so that $H = X(A)$. Then $A \cap S \neq \emptyset$ (indeed $|A \cap S| = \tau$ by Lemma 3.7(iii)) so there are $\tau$-many pairs $(\gamma, \eta)$ such that $D_\eta^\gamma \subseteq X(A) \cap X(S)$. Each such $D_\eta^\gamma$ meets $U$, so

$|X(S) \cap (H \cap U)| = |X(S) \cap X(A) \cap U| = \tau.$

For (ii), let $p \in X \backslash X(S \cap S')$, say $p \in D_\eta^\gamma$, and using Lemma 3.7(iv) choose $t = (x, \xi) \in Z \times 2^\tau$ such that $\eta \in A_t^1$ and $S \cap S' \subseteq A_t^0$. Then $p \in X(A_t^1) = H_t^1$ and $X(S \cap S') \subseteq X(A_t^0) = H_t^0$, so $H_t^1$ is a $\mathcal{U}$-open neighborhood of $p$ disjoint from $X(S \cap S')$. Thus $X(S \cap S')$ is closed in $(X,\mathcal{U})$.

Given $\emptyset \neq U \in \mathcal{T}$ and $H = \bigcap_{t\in F} H_t^{f(t)}$ as in (a) set $A := \bigcap_{t\in F} A_t^{f(t)}$, so that $H = X(A)$, and note from Lemma 3.7(iii) that $|A \cap (S' \backslash S)| = \tau$. Then

$U \cap X(A) \cap X(S' \backslash S) = U \cap H \cap X(S' \backslash S) \neq \emptyset,$

so $X(S' \backslash S) = X(S') \backslash X(S)$ is dense in $(X,\mathcal{U})$. *A fortiori* $X \backslash X(S)$ is dense in $(X,\mathcal{U})$, so the closed set $X(S)$ is nowhere dense in $(X,\mathcal{U})$. $\square$

Now we are ready to prove the case $|X| = \Delta(X)$ of items $(i = 4)$ and $(i = 5)$ of the Abstract.

**Theorem 3.9** *Let $X = (X,\mathcal{T})$ be a crowded, maximally resolvable Tychonoff space with $S(X,\mathcal{T}) \leq |X| = \Delta(X,\mathcal{T}) = \kappa$. Then there is a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) *$S(X,\mathcal{U}) = S(X,\mathcal{T})$ and $\Delta(X,\mathcal{U}) = \Delta(X,\mathcal{T})$;*

(b) *$(X,\mathcal{U})$ is extraresolvable; and*

(c) *$(X,\mathcal{U})$ is not maximally resolvable.*

Proof. The topology $\mathcal{U}$ will be of the form $\mathcal{U} = \mathcal{T}_{\mathcal{KAD}}$. We first define the families $\mathcal{K}$, $\mathcal{A}$ and $\mathcal{D}$.

Let $\mathcal{D} = \{D_\eta : \eta < \kappa\}$ be a dense partition of $(X,\mathcal{T})$ which witnesses the maximal resolvability of $(X,\mathcal{T})$. (Note. To match the notation used throughout Section 2, more formally we take $\tau = 1 = \{0\}$ and $D_\eta = D_\eta^0$ in Notation 2.1; then $X(S) = \bigcup_{\eta\in S} D_\eta$ for $S \subseteq \kappa$.)

Let $\mathcal{A} = \{\mathcal{A}_\xi : \xi < 2^\kappa\}$ with $\mathcal{A}_\xi = \{A_\xi^0, A_\xi^1\}$ and $\mathcal{S}_{er} \subseteq \mathcal{P}(\kappa)$ as given in Lemma 3.7, and re-index $\mathcal{A}$ in the form $\mathcal{A} = \{\mathcal{A}_t : t \in X \times 2^\kappa\}$. We partition the set $2^\kappa$ in the form $2^\kappa = T_0 \cup T_1$ with $|T_0| = |T_1| = 2^\kappa$. We assume without loss of generality that the families $\{\mathcal{A}_t : t \in X \times T_1\}$ and $\mathcal{S}_{er}$ satisfy conditions (i) through (iv) of Lemma 3.7.

The definition of the family $\mathcal{K}$ parallels the construction in Definition 2.9, but with modifications. Specifically:

Let $\mathcal{M} = \{M_\xi : \xi < 2^\kappa\} = \mathcal{P}(X)$ with $M_0 = \emptyset$ and define $\widetilde{M} = \{\widetilde{M_\xi} : \xi < 2^\kappa\}$ as follows.

$\widetilde{M_0} = \emptyset$; and
if $0 < \xi < 2^\kappa$ and $\widetilde{M_\eta}$ has been defined for all $\eta < \xi$ then
$\widetilde{M_\xi} = M_\xi$ if each set of the form
$(M_\xi \cup \widetilde{M_{\eta_0}} \cup \widetilde{M_{\eta_1}} \cup \cdots \cup \widetilde{M_{\eta_n}}) \cap X(S)$ $(n < \omega,\ \eta_i < \xi,\ S \in \mathcal{S}_{er})$
has nonempty interior in the space $(X(S), \mathcal{T}^{\mathcal{AD}})$
$= \emptyset$ otherwise.

Then with $T_0, T_1 \subseteq 2^\kappa$ as above, we write $\mathcal{K} = \mathcal{K}_0 \cup \mathcal{K}_1$ with $\mathcal{K}_i = \{K_\xi : \xi \in T_i\}$; we arrange that $\{K_\xi : \xi \in T_0\}$ is a faithful indexing of $\widetilde{\mathcal{M}}$, and $K_\xi = \emptyset$ for each $\xi \in T_1$.

We claim that the topology $\mathcal{U} = \mathcal{T}_{\mathcal{KAD}}$ is as required.

We verify conditions (a), (b) and (c). Indeed as to (c) we will show that $(X, \mathcal{U})$ is not even $S(X, \mathcal{T})$-resolvable.

(a) From Corollary 2.6 and Lemma 2.7 we have

$$\kappa = \Delta(X, \mathcal{T}) = \Delta(X, \mathcal{T}_{\mathcal{KAD}}) = \Delta(X, \mathcal{U}) \text{ and } \kappa = S(X, \mathcal{T}) = S(X, \mathcal{T}_{\mathcal{KAD}}) = S(X, \mathcal{U}).$$

(b) It is enough to show that

(i) if $S \in \mathcal{S}_{er}$ then $X(S)$ is dense in $(X, \mathcal{U})$; and

(ii) if $S$ and $S'$ are distinct elements of $\mathcal{S}_{er}$, then $X(S) \cap X(S')$ is (closed and) nowhere dense in $(X, \mathcal{U})$.

For (i), we must show that if $\emptyset \neq U \in \mathcal{T}$ and $W = \bigcap_{t \in F} H_t^{f(t)} \backslash K_t \in \mathcal{T}_{\mathcal{KAD}}$ with $F \in [X \times 2^\kappa]^{<\omega}$ and $f \in \{0,1\}^F$, then $X(S) \cap (U \cap W) \neq \emptyset$. For that, set $A := \bigcap_{t \in F} A_t^{f(t)}$, so that $H := \bigcap_{t \in F} H_t^{f(t)} = X(A)$ and $W = X(A) \backslash K$ with $K = \bigcup\{K_t : t \in F\}$. Since $\mathrm{int}_{(X(S), \mathcal{T}^{\mathcal{AD}})}(K \cap X(S)) = \emptyset$, the set $X(S) \backslash K$ is dense in $(X(S), \mathcal{T}^{\mathcal{AD}})$. From Lemma 3.7(iii) we have $A \cap S \neq \emptyset$, so $X(A) \cap U \cap X(S) = X(A \cap S) \cap U \neq \emptyset$. Since $\emptyset \neq X(A) \cap U \in \mathcal{T}^{\mathcal{AD}}$ and $K$ is closed and discrete in $\mathcal{U}$, we have $(X(S) \backslash K) \cap X(A) \cap U \neq \emptyset$ and therefore $X(S) \cap W \cap U \neq \emptyset$, as required.

Before verifying (ii), we show this for later use.

each set $\bigcup_{\eta \in G} D_\eta$ with $G \in [\kappa]^{<\omega}$ is in $\mathcal{K}$. (1)

If that fails, there are $U \in \mathcal{T}$, $K \in \mathcal{K}$, $S \in \mathcal{S}_{er}$, and $H \in \mathcal{T}^{\mathcal{AD}}$ such that

$U \cap H \cap X(S) \subseteq [(\bigcup_{\eta \in G} D_\eta) \cup K] \cap X(S)$.

Here $H = X(A)$ with $A = \bigcup_{t \in F} A_t^{f(t)}$. Since $\{\mathcal{A}_t : t \in X \times T_1\}$ has the strong small-set-separating property, for each $\eta \in G$ there are infinitely many indices $v_\eta$ such that ($\mathcal{A}_{v_\eta} \in \mathcal{A}$ and) $\mathcal{A}_{v_\eta}$ separates $\{\eta\}$ and $\emptyset$. For each $\eta \in G$ we choose such $v_\eta$ such that $v_\eta \notin F$, so that $A \cap \bigcap_{\eta \in G} A_{v_\eta}^1 \neq \emptyset$. We set $H' := \bigcap_{\eta \in G} X(A_{v_\eta}^1)$, so that $\emptyset \neq U \cap H \cap H' \in \mathcal{T}^{\mathcal{AD}}$. Then

$[(\bigcup_{\eta \in G} D_\eta) \cup K] \cap X(S) \supseteq U \cap H \cap X(S) \supseteq U \cap H \cap H' \cap X(S) \neq \emptyset$.

Here $U \cap H \cap H' \cap X(S) \neq \emptyset$ because $X(S)$ is dense in $\mathcal{U}$ (see (b)(i)) and $H'$ differs from certain $W' \in \mathcal{U}$ by a $\mathcal{U}$-closed, $\mathcal{U}$-discrete set $K \in \mathcal{K}$.

Since $(\bigcap_{\eta \in G} A_{v_\eta}^1) \cap (\bigcap_{\eta \in F} D_\eta) = \emptyset$, we then have

$K \cap X(S) \supseteq U \cap H \cap H' \cap X(S) \neq \emptyset$,

contradicting the condition $K \in \mathcal{K}$. Thus (1) is shown.

Now for (ii), let $x \in X \backslash X(S \cap S')$, say with $x \in D_\eta$, and using Lemma 3.7(iv)(a) choose $u \in X \times T_1$ such that $\eta \in A_u^1$ and $S \cap S' \subseteq A_u^0$. Then $x \in X(A_u^1) = H_u^1 = W_u^1$ and

$X(S) \cap X(S') \subseteq X(A^0_u) = H^0_u = W^0_u$ (since $K_\xi = \emptyset$ for $\xi \in T_1$ in Definition 2.2), so $W^0_u$ is a neighborhood in $(X, \mathcal{U})$ of $x$ which is disjoint from $X(S \cap S')$. Thus $X(S \cap S')$ is closed in $(X, \mathcal{U})$.

To see that the closed set $X(S \cap S')$ is nowhere dense in $(X, \mathcal{U})$, suppose (taking notation as above) that there are nonempty $U \in \mathcal{T}$ and $W = X(A) \backslash K \in \mathcal{U}$ with $A = \bigcap_{t \in F} A^{f(t)}_t$ such that $U \cap W \subseteq X(S \cap S')$. Fix $\eta \in S \cap S'$ and use Lemma 3.7(iv)(b) to find $u \in X \times T_1$ such that $u \notin F$ and $(S \cap S') \cap A^1_u = \{\eta\}$. Then $X(A^1_u) \cap X(S \cap S') = D_\eta$, and the condition $u \notin F$ implies $\emptyset \neq X(A) \cap X(A^1_u) \in \mathcal{T}^{\mathcal{AD}}$, which further implies $\emptyset \neq U \cap W \cap X(A^1_u) \in \mathcal{U}$. Hence $U \cap W \cap X(A^1_u) \subseteq U \cap W \subseteq X(S \cap S')$ and from (1) we have

$$\emptyset \neq U \cap W \cap X(A^1_u) \subseteq X(A^1_u) \cap X(S \cap S') \subseteq D_\eta \cup K \in \mathcal{K}.$$

But from Lemma 2.5 the space $D_\eta \cup K$ is closed and discrete in $(X, \mathcal{U})$, a contradiction. The proof of (b) is complete.

(c) Here we show more, namely that $(X, \mathcal{U})$ is not even $S(X, \mathcal{T})$-resolvable. Arguing much as in Theorem 2.11(b), we first show this:

$$\text{if } \xi < 2^\kappa \text{ and } \mathrm{int}_{(X(S),\mathcal{U})}(M_\xi \cap X(S)) = \emptyset \text{ for all } S \in \mathcal{S}_{er}, \text{ then } \widetilde{M_\xi} = M_\xi \in \mathcal{K}. \quad (2)$$

For that, we must show for fixed $S \in \mathcal{S}_{er}$ and fixed $K \in \mathcal{K}$ that

$$\mathrm{int}_{(X(S),\mathcal{T}^{\mathcal{AD}})}[(M_\xi \cup K) \cap X(S)] = \emptyset.$$

To see that, let $\emptyset \neq U \in \mathcal{T}$ and $W = \bigcap_{t \in F} W^{f(t)}_t \in \mathcal{U}$. Since $X(S) \setminus M_\xi$ is dense in $(X(S), \mathcal{U})$ and $\emptyset \neq U \cap W \in \mathcal{U}$, we have that $Y := (X(S) \backslash M_\xi) \cap (U \cap W)$ is dense in $(X(S) \cap (U \cap W), \mathcal{U})$. Thus $Y$ is crowded, so since $W$ differs from $H := \bigcap_{t \in F} H^{f(t)}_t \in \mathcal{T}^{\mathcal{AD}}$ by a set $K' \in \mathcal{K}$ we have from Lemma 2.10 that $Y \backslash (K' \cup K)$ remains dense in $(X(S) \cap (U \cap W), \mathcal{U})$, hence dense in $(X(S) \cap (U \cap H), \mathcal{T}^{\mathcal{AD}})$. Thus

$$\mathrm{int}_{(X(S),\mathcal{T}^{\mathcal{AD}})}[(M_\xi \cup K) \cap X(S)] = \emptyset,$$

as required, and (2) is proved.

To complete the proof of (c) we argue by contradiction, supposing that $\{E_\eta : \eta < S(X, \mathcal{T})\}$ is a pairwise disjoint family of dense subsets of $(X, \mathcal{U})$. For each $\eta < S(X, \mathcal{T})$ there is $S_\eta \in \mathcal{S}_{er}$ such that $\mathrm{int}_{(X(S_\eta),\mathcal{U})}(E_\eta \cap X(S_\eta)) \neq \emptyset$, so there are nonempty $U_\eta \in \mathcal{T}$ and $W_\eta = X(A_\eta) \backslash K_\eta \in \mathcal{U}$ with $A_\eta = \bigcap_{t \in F_\eta} A^{f_\eta(t)}_t$ such that

$$\emptyset \neq U_\eta \cap W_\eta \cap X(S_\eta) \subseteq E_\eta \cap X(S_\eta).$$

For notational simplicity set $V_\eta := U_\eta \cap W_\eta \cap X(S_\eta)$ for $\eta < S(X, \mathcal{T})$. Then

$$\{V_\eta : \eta < S(X, \mathcal{T}\} \text{ is a pairwise disjoint family,} \quad (3)$$

since $V_\eta \subseteq E_\eta$ and $\{E_\eta : \eta < S(X, \mathcal{T}\}$ is pairwise disjoint.

Now recall, using the notation $\mathcal{J}$, $\mathcal{W}$, $\mathcal{H}_t$, $\mathcal{L}$, $\mathcal{D}$, $J$, $W$, $H$, $K$, $F$ and $L$ as in (the proof of) Lemma 3.7, that each of the present sets $S_\eta$ is of the form $L^0_\zeta \in \mathcal{L}_\zeta \in \mathcal{L}$ for some $\zeta < 2^\kappa$, and $A_\eta$ is a Boolean combination of sets from $\mathcal{J} \cup \mathcal{W}$, say $A_\eta = J \cap W$ where $J, W$ are as in the proof of Lemma 3.7. Write $W = H \setminus K$ with $H$ as in Lemma 3.7 and with $K$ of the form $K = \bigcup_{i<n} (\bigcap_{\zeta \in F_i} L^0_\zeta)$ with $1 < |F_i| < \omega$. For each $i < n$ choose $\zeta_i \in F_i$ such that $L^0_{\zeta_i} \neq S_\eta$. Then $L := \bigcap_{i<n} L^1_{\zeta_i}$ satisfies $L \cap K = \emptyset$ and $L \cap S_\eta \neq \emptyset$.

Since $\mathcal{J} \cup \mathcal{L} \cup \{\mathcal{D}\}$ is an independent family and the elements of each partition in $\mathcal{H}$ are unions of some dense sets in $\mathcal{D}$, the family $\mathcal{J} \cup \mathcal{H} \cup \mathcal{L}$ is also an independent family. Since $J$ is a Boolean combination of sets from $\mathcal{J}$, $H$ is a Boolean combination of sets from $\mathcal{H}$, and $L \cap S_\eta$ is a Boolean combination of sets from $\mathcal{L}$, we have $J \cap H \cap L \cap S_\eta \neq \emptyset$. Since $W = H \setminus K$ and $L \cap K = \emptyset$, we have $H \cap L \subseteq W$ and

$$\emptyset \neq J \cap H \cap L \cap S_\eta = J \cap (H \cap L) \cap S_\eta \subseteq J \cap W \cap S_\eta = A_\eta \cap S_\eta.$$

This argument shows that for each $\eta < S(X, \mathcal{T})$ there is a Boolean combination $N_\eta$ of sets from the independent family $\mathcal{J} \cup \mathcal{H} \cup \mathcal{L}$, of the form $N_\eta = P \cap H \cap L \cap S_\eta$, such that

$$\emptyset \neq N_\eta \subseteq A_\eta \cap S_\eta. \tag{4}$$

For simplicity write $\mathcal{B} := \mathcal{J} \cup \mathcal{H} \cup \mathcal{L} = \{\mathcal{B}_t : t \in T\}$ with $|T| = 2^\kappa$ and write each $N_\eta$ in the form $N_\eta = \bigcap_{t \in F_\eta} B_t^{i_\eta(t)}$ with $F_\eta \in [T]^{<\omega}$, $i_\eta \in \{0,1\}^{F_\eta}$. Since $S(X, \mathcal{T})$ is a regular cardinal there are, by the Erdős-Rado theorem on quasi-disjoint sets [11], [12] (the "$\Delta$-system Lemma" [24]) a (finite) set $F$ and $Q \subseteq S(X, \mathcal{T})$ with $|Q| = S(X, \mathcal{T})$ such that $F_\eta \cap F_{\eta'} = F$ whenever $\eta, \eta' \in Q$, $\eta \neq \eta'$. We assume without loss of generality that $F \neq \emptyset$ and that $i_\eta(t) = i_{\eta'}(t) \in \{0, 1\}$ for all $\eta, \eta' \in Q$, $t \in F$. Then

$$\emptyset \neq N_\eta \cap N_{\eta'} \subseteq (A_\eta \cap S_\eta) \cap (A_{\eta'} \cap S_{\eta'})$$

for distinct $\eta, \eta' \in Q$.

Since $\emptyset \neq U_\eta \in \mathcal{T}$ for each $\eta \in Q$, there are distinct $\eta_0, \eta_1 \in Q$ (henceforth fixed) such that $U_{\eta_0} \cap U_{\eta_1} \neq \emptyset$.

Now $\emptyset \neq U_{\eta_k} \cap X(A_{\eta_k}) \in \mathcal{T}^{\mathcal{AD}}$, and $X(S_{\eta_k})$ is dense in $(X, \mathcal{U})$, and $K_{\eta_k}$ is closed and nowhere dense in $(X, \mathcal{U})$, so from $U_{\eta_0} \cap U_{\eta_1} \neq \emptyset$ follows

$$[U_{\eta_0} \cap X(A_{\eta_0}) \cap X(S_{\eta_0}) \backslash K_{\eta_0}] \cap [U_{\eta_1} \cap X(A_{\eta_1}) \cap X(S_{\eta_1}) \backslash K_{\eta_1}] \neq \emptyset,$$

that is:

$$V_{\eta_0} \cap V_{\eta_1} = [U_{\eta_0} \cap W_{\eta_0} \cap X_{\eta_0}] \cap [U_{\eta_1} \cap W_{\eta_1} \cap X_{\eta_1}] \neq \emptyset,$$

which contradicts (3). $\square$

**Theorem 3.10** *Let $X = (X, \mathcal{T})$ be a crowded, maximally resolvable Tychonoff space with $S(X, \mathcal{T}) \leq |X| = \Delta(X, \mathcal{T}) = \kappa$. Then there is a Tychonoff refinement $\mathcal{U}$ of $\mathcal{T}$ such that*

(a) $S(X, \mathcal{U}) = S(X, \mathcal{T})$ *and* $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T})$*;*

(b) $(X, \mathcal{U})$ *is maximally resolvable;*

(c) $(X, \mathcal{U})$ *is extraresolvable; and*

(d) $(X, \mathcal{U})$ *is not strongly extraresolvable.*

Proof. We expand in three steps with (modified) $\mathcal{KID}$-like expansions $\mathcal{T} \subseteq \mathcal{T}' \subseteq \mathcal{T}'' \subseteq \mathcal{U}$. Here are the details.

Step 1. Let $\mathcal{D}_1 = \{D_\eta^\gamma : \eta < \kappa, \gamma < \kappa\}$ be a partition of $X$ into $\mathcal{T}$-dense subsets, let $\mathcal{I}_1 = \{\mathcal{I}_t : t \in X \times 2^\kappa\}$ be a $\kappa$-independent family of partitions of $X$ with the strong small-set-separating property with each $\kappa_t < S(X, \mathcal{T})$, and let $\mathcal{K}_1 := \{K_\xi : \xi < 2^\kappa\} = [X]^{<\kappa}$ (with repetitions permitted in the indexing of $\mathcal{K}_1$). Clearly conditions (1), (2), (3) and (4) of 2.4 are satisfied (with $Z = X$). To see that (5) also is satisfied, fix nonempty $U \in \mathcal{T}$ and $H = X(I)$ with $F \in [X \times 2^\kappa]^{<\omega}$, $f \in \Pi_{t \in F}\, \kappa_t$, and $I := \bigcap_{t \in F} I_t^{f(t)}$. We have $|I| = \kappa$, so $H \cap D^\gamma \supseteq D_\eta^\gamma$ for $\kappa$-many $\eta < \kappa$, each dense in $(X, \mathcal{T})$, so $|D \cap U \cap H| = \kappa$. Thus $D^\gamma \cap U \cap H \subseteq K_\xi \cap D^\gamma$ is impossible, so (5) holds. It follows that $\mathcal{T}' := \mathcal{T}_{\mathcal{K}_1 \mathcal{I}_1 \mathcal{D}_1}$ has the properties given in Lemma 2.5, in particular each $K_\xi \in \mathcal{K}_1 = [X]^{<\kappa}$ is closed and discrete in $(X, \mathcal{T}')$, hence nowhere dense, so $\mathrm{nwd}(X, \mathcal{T}') = \kappa$.

Step 2. Apply Theorem 3.6 (to the space $(X, \mathcal{T}')$) to find an expansion $\mathcal{T}'' \supseteq \mathcal{T}'$ (with $\mathcal{T}''$ of the form $\mathcal{T}'' = \mathcal{T}'_{\mathcal{KID}}$) such that $S(X, \mathcal{T}'') = S(X, \mathcal{T}')$, $\Delta(X, \mathcal{T}'') = \Delta(X, \mathcal{T}')$, and $(X, \mathcal{T}'')$ is maximally resolvable but not extraresolvable.

Step 3. By Theorem 3.8 with $\tau = \kappa$ and with $\mathcal{T}''$ replacing $\mathcal{T}$ there, there is an expansion $\mathcal{U} \supseteq \mathcal{T}''$ (with $\mathcal{U}$ of the form $\mathcal{T}''^{\mathcal{AD}}$) such that

$$S(X,\mathcal{U}) = S(X,\mathcal{T}'') = S(X,\mathcal{T}) \text{ and } \Delta(X,\mathcal{U}) = \Delta(X,\mathcal{T}'') = \Delta(X,\mathcal{T})$$

and such that $(X,\mathcal{U})$ is maximally resolvable and $2^\kappa$-extraresolvable. Furthermore each set $K \in [X]^{<\kappa}$ is closed and discrete in $(X,\mathcal{T}')$, hence in $(X,\mathcal{T}'')$, so $\mathrm{nwd}(X,\mathcal{T}'') = \kappa$. Thus any family $\mathcal{E}$ (with $|\mathcal{E}| > \Delta(X,\mathcal{U}) = \Delta(X,\mathcal{T}')$) witnessing the strong extraresolvability of $(X,\mathcal{U})$ would witness the strong extraresolvability of $(X,\mathcal{T}'')$, contrary to the fact that $(X,\mathcal{T}'')$ is not (even) extraresolvable. □

# 4 The General Case

The five principal results proved in Section 3 require, in addition to the essential overarching hypothesis $S(X,\mathcal{T}) \leq \Delta(X,\mathcal{T})$, also the artificial condition $|X| = \Delta(X,\mathcal{T})$. Since for each of those five results it is essentially the same argument which allows us to pass from the special case ($|X| = \Delta(X,\mathcal{T})$) to the unrestricted case ($|X|$ is arbitrary), we corral all five of the general results into one extended statement. Theorem 4.2, then, duplicates the essentials of our Abstract.

**Lemma 4.1** *Let $(X,\mathcal{T})$ be a crowded Tychonoff space. For $\emptyset \neq U \in \mathcal{T}$ there is $V \in \mathcal{T}$ such that $K := \mathrm{cl}_{(X,\mathcal{T})} V$ satisfies $V \subseteq K \subseteq U$ and $\Delta(U) = \Delta(K) = |K|$.*

Proof. Choose $W \in \mathcal{T}$ such that $W \subseteq U$ and $|W| = \Delta(U)$, and choose $V \in \mathcal{T}$ so that $V \neq \emptyset$ and $V \subseteq K := \mathrm{cl}_{(X,\mathcal{T})} V \subseteq W$. □

**Theorem 4.2** *Let $(X,\mathcal{T})$ be a crowded, maximally resolvable Tychonoff space such that $S(X,\mathcal{T}) \leq \Delta(X,\mathcal{T}) = \kappa$. Then there are Tychonoff expansions $\mathcal{U}_i$ $(1 \leq i \leq 5)$ of $\mathcal{T}$, with $\Delta(X,\mathcal{U}_i) = \Delta(X,\mathcal{T})$ and $S(X,\mathcal{U}_i) \leq \Delta(X,\mathcal{U}_i)$, such that $(X,\mathcal{U}_i)$ is:*

*$(i = 1)$ $\omega$-resolvable but not maximally resolvable;*

*$(i = 2)$ [if $\kappa'$ is regular, with $S(X,\mathcal{T}) \leq \kappa' \leq \kappa$] $\tau$-resolvable for all $\tau < \kappa'$, but not $\kappa'$-resolvable;*

*$(i = 3)$ maximally resolvable, but not extraresolvable;*

*$(i = 4)$ extraresolvable, but not maximally resolvable;*

*$(i = 5)$ maximally resolvable and extraresolvable, but not strongly extraresolvable.*

Proof. (Recall our frequently used convention that when $(X,\mathcal{T})$ is a space and $Y \subseteq X$, the symbol $(Y,\mathcal{T})$ denotes the set $Y$ with the topology inherited from $(X,\mathcal{T})$.)

Using Lemma 4.1 (with $U = X$), choose a regular-closed set $X' \subseteq X$ such that

$$S(X',\mathcal{T}) \leq S(X,\mathcal{T}) \leq \Delta(X,\mathcal{T}) = \Delta(X',\mathcal{T}) = |X'| = \kappa.$$

The definition of the topologies $\mathcal{U}_i$ for $i = 1, 2, 3$, and the verification that they are as required, will be straightforward. We discuss these first, leaving the cases $(i = 4, 5)$ for treatment later in the proof.

The space $(X',\mathcal{T})$ satisfies the hypotheses of Theorems 3.1, 3.3, 3.6, so there are Tychonoff expansions $\mathcal{U}_i'$ $(i = 1, 2, 3)$ of $\mathcal{T}$ on $X'$ satisfying their respective conclusions. Let $\mathcal{U}_i$ $(i = 1, 2, 3)$ be the topology on $X$ for which $(X',\mathcal{U}_i')$ and $(X\backslash X',\mathcal{T})$ are open-and-closed subspaces of $(X,\mathcal{U}_i)$. It is easily seen that $(X,\mathcal{U}_i)$ is a Tychonoff space. Further we

have $\mathcal{T} \subseteq \mathcal{U}_i$, since if $U \in \mathcal{T}$ then $U \cap X'$ is open in $(X', \mathcal{T})$, hence in $(X', \mathcal{U}'_i)$, hence in $(X', \mathcal{U}_i)$, and $U \cap (X \backslash X')$ is open in $(X \backslash X', \mathcal{T}) = (X \backslash X', \mathcal{U}_i)$.

For $i = 1, 2, 3$ we have, using $\Delta(X', \mathcal{U}'_i) = \Delta(X', \mathcal{T}) \leq \Delta(X \backslash X', \mathcal{T})$, that

$\Delta(X, \mathcal{U}_i) = \min\{\Delta(X', \mathcal{U}_i), \Delta(X \backslash X', \mathcal{U}_i)\} = \Delta(X', \mathcal{T}) = \Delta(X, \mathcal{T}) = \kappa$.

Further for $i = 1, 3$ we have, using $S(X', \mathcal{U}'_i) = S(X', \mathcal{T})$, that

$S(X, \mathcal{U}_i) = S(X', \mathcal{U}_i) + S(X \backslash X' \mathcal{U}_i) = S(X', \mathcal{T}) + S(X \backslash X', \mathcal{T}) = S(X, \mathcal{T})$,

while for $i = 2$ we have

$S(X, \mathcal{U}_2) = S(X', \mathcal{U}_2) + S(X \backslash X', \mathcal{U}_2) = \kappa' + S(X \backslash X', \mathcal{U}_2) = \kappa' + S(X \backslash X', \mathcal{T}) = \kappa'$.

We verify the required (non-) resolvability properties of the spaces $(X, \mathcal{U}_i)$ for $i = 1, 2, 3$.

In each case, $(X, \mathcal{U}_i)$ is the union of two disjoint open-and-closed subspaces, namely $(X', \mathcal{U}_i)$ and $(X \backslash X', \mathcal{U}_i) = (X \backslash X', \mathcal{T})$. When $i = 1$, these are both $\omega$-resolvable; when $i = 2$, both are $\tau$-resolvable for each $\tau < \kappa'$; when $i = 3$, both are $\kappa$-resolvable. Thus $(X, \mathcal{U}_1)$ is $\omega$-resolvable; $(X, \mathcal{U}_2)$ is $\tau$-resolvable for all $\tau < \kappa'$; and $(X, \mathcal{U}_3)$ is $\kappa$-resolvable (i.e., is maximally resolvable).

Since $(X', \mathcal{U}'_1) = (X', \mathcal{U}_1)$ is open in $(X, \mathcal{U}_1)$ and is not $\Delta(X, \mathcal{T})$-resolvable, surely $(X, \mathcal{U}_1)$ is not $\Delta(X, \mathcal{T})$-resolvable, i.e., is not $\Delta(X', \mathcal{U}_1)$-resolvable.

The space $(X, \mathcal{U}_2)$ is not $\kappa'$-resolvable, since its open subspace $(X', \mathcal{U}'_2) = (X', \mathcal{U}_2)$ is not $\kappa'$-resolvable.

The space $(X, \mathcal{U}_3)$ is not extraresolvable, since its open subspace $(X', \mathcal{U}'_3) = (X', \mathcal{U}_3)$ is not extraresolvable (and satisfies $\Delta(X', \mathcal{U}'_3) = \Delta(X, \mathcal{U}_3)$).

We turn to the cases $(i = 4, 5)$.

Let $\mathcal{V} \subseteq \mathcal{T}$ be chosen maximal with respect to the properties

$\{\mathrm{cl}_{(X,\mathcal{T})} V : V \in \mathcal{V}\}$ is pairwise disjoint, and

$|V| = |\mathrm{cl}_{(X,\mathcal{T})} V| = \Delta(V)$ for each $V \in \mathcal{V}$.

We write $\mathcal{V} = \{V_\beta : \beta < \alpha\}$ and $X'_\beta := \mathrm{cl}_{(X,\mathcal{T})} V_\beta$, the indexing chosen with $V_0$ and $X'_0 = X'$ as in the first part of this proof: $|X'_0| = \Delta(X'_0, \mathcal{T}) = \Delta(X, \mathcal{T})$.

The space $(X'_0, \mathcal{T})$ satisfies

$S(X'_0, \mathcal{T}) \leq S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T}) = \Delta(X'_0, \mathcal{T}) = |X_0| = \kappa$,

so by Theorems 3.9 and 3.10 there are Tychonoff refinements $\mathcal{U}'_{0,4}$ and $\mathcal{U}'_{0,5}$ of $(X'_0, \mathcal{T})$, with

$S(X'_0, \mathcal{U}'_{0,4}) = S(X'_0, \mathcal{U}'_{0,5}) = S(X'_0, \mathcal{T})$ and

$\Delta(X'_0, \mathcal{U}'_{0,4}) = \Delta(X'_0, \mathcal{U}'_{0,5}) = \Delta(X'_0, \mathcal{T}) = \kappa$,

such that

$(X'_0, \mathcal{U}'_{0,4})$ is extraresolvable, but not maximally resolvable; and

$(X'_0, \mathcal{U}'_{0,5})$ is maximally resolvable and extraresolvable, but not strongly extraresolvable.

For $0 < \beta < \alpha$ the spaces $(X'_\beta, \mathcal{T})$ satisfy

$S(X'_\beta, \mathcal{T}) \leq S(X, \mathcal{T}) \leq \kappa = \Delta(X, \mathcal{T}) \leq \Delta(X'_\beta, \mathcal{T}) = |X'_\beta|$.

By Theorem 3.8, taking $\tau = \kappa_\beta := |X'_\beta|$ there, there are for $0 < \beta < \alpha$ Tychonoff expansions $\mathcal{U}'_\beta$ of $(X'_\beta, \mathcal{T})$ such that

$S(X'_\beta, \mathcal{U}'_\beta) = S(X'_\beta, \mathcal{T})$ and $\Delta(X'_\beta, \mathcal{U}'_\beta) = \Delta(X'_\beta, \mathcal{T})$,

and $(X'_\beta, \mathcal{U}'_\beta)$ is $\kappa_\beta$-resolvable and $2^{\kappa_\beta}$-extraresolvable. Then since $\kappa \leq \kappa_\beta$, the space $(X'_\beta, \mathcal{U}'_\beta)$ is $\kappa$-resolvable and $2^\kappa$-extraresolvable.

Now for $(i = 4, 5)$ we define $\mathcal{U}_i$ to be the smallest topology on $X$ such that

(1) $\mathcal{T} \subseteq \mathcal{U}_i$,

(2) $(X'_0, \mathcal{U}'_{0,i})$ is open-and-closed in $(X, \mathcal{U}_i)$, and

(3) each space $(X'_\beta, \mathcal{U}'_\beta)$ (with $0 < \beta < \alpha$) is open-and-closed in $(X, \mathcal{U}_i)$.

To see that $(X, \mathcal{U}_i)$ is a Tychonoff space, it is enough to note that if $x \in \bigcup_{\beta<\alpha} X'_\beta$, say $x \in X'_\beta$, then $X'_\beta$ is an open Tychonoff neighborhood of $x$ in $(X, \mathcal{U}_i)$; while if $x \notin \bigcup_{\beta<\alpha} X'_\beta$, then the $\mathcal{T}$-open neighborhoods of $x$ remain basic at $x$ in $(X, \mathcal{U}_i)$ (so if $x \in U \in \mathcal{U}_i$ then there is a $\mathcal{U}_i$-continuous (even, $\mathcal{T}$-continuous) real-valued function $f$ on $X$ such that $f(x) = 0$ and $f = 1$ on $X \backslash U$).

For $\beta < \alpha$ we have

$\Delta(X'_\beta, \mathcal{U}_i) = \Delta(X'_\beta, \mathcal{U}'_\beta) = \Delta(X'_\beta, \mathcal{T}) \geq \Delta(X'_0, \mathcal{T})$,

so $\Delta(X, \mathcal{U}_i) = \min_{\beta<\alpha} \Delta(X'_\beta, \mathcal{U}_i) = \Delta(X'_0, \mathcal{T}) = \Delta(X, \mathcal{T}) = \kappa$.

We verify for $(i = 4, 5)$ that $S(X, \mathcal{U}_i) \leq \Delta(X, \mathcal{U}_i)$. For a cellular family $\mathcal{W} \subseteq \mathcal{U}_i$ and $\beta < \alpha$ let $\mathcal{W}(\beta) := \{W \cap X'_\beta : W \in \mathcal{W}, W \cap X'_\beta \neq \emptyset\}$. Then $\mathcal{W}(\beta)$ is a cellular family by Lemma 4.1. The set $\bigcup_{\beta<\alpha} X'_\beta$ is dense in $(X, \mathcal{T})$, so $\mathcal{W} = \bigcup_{\beta<\alpha} \mathcal{W}(\beta)$, so $|\mathcal{W}| \leq \Sigma_{\beta<\alpha} |\mathcal{W}(\beta)|$ with each

$|\mathcal{W}(\beta)| < S(X'_\beta, \mathcal{U}'_\beta) = S(X'_\beta, \mathcal{T}) \leq S(X, \mathcal{T})$.

Since $\alpha < S(X, \mathcal{T})$ and $S(X, \mathcal{T})$ is regular, we have $|\mathcal{W}| < S(X, \mathcal{T})$. It follows that

$S(X, \mathcal{U}_i) \leq S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T}) = \Delta(X, \mathcal{U}_i)$.

It remains to verify that the spaces $(X, \mathcal{U}_4)$ and $(X, \mathcal{U}_5)$ have the required (non-) resolvability properties.

Each space $(X'_\beta, \mathcal{U}_4)$ is open in $(X, \mathcal{U}_4)$, with $\bigcup_{\beta<\alpha} X'_\beta$ dense in $(X, \mathcal{U}_4)$. Each space $(X'_\beta, \mathcal{U}_4)$ is extraresolvable (by Theorem 3.9(b) for $\beta = 0$, by Theorem 3.8 for $0 < \beta < \alpha$), so for each $\beta < \alpha$ there is a family $\mathcal{E}_\beta = \{E_\beta(\eta) : \eta < \kappa^+\}$ of dense subsets of $(X'_\beta, \mathcal{U}_4)$ such that $E_\beta(\eta) \cap E_\beta(\eta')$ is nowhere dense in $(X'_\beta, \mathcal{U}_4)$ whenever $\eta < \eta' < \kappa^+$. Then with $E(\eta) := \bigcup_{\beta<\alpha} E_\beta(\eta)$, the family $\{E(\eta) : \eta < \kappa^+\}$ witnesses the extraresolvability of $(X, \mathcal{U}_4)$. The space $(X, \mathcal{U}_4)$ is not maximally resolvable (i.e., is not $\kappa$-resolvable), however, since its open subspace $(X'_0, \mathcal{U}_4) = (X'_0, \mathcal{U}_{4,0})$ is not $\kappa$-resolvable.

The space $\bigcup_{\beta<\alpha} X'_\beta$ is open and dense in $(X, \mathcal{U}_5)$, with each $(X'_\beta, \mathcal{U}_5)$ open and $\kappa$-resolvable and $2^\kappa$-extraresolvable, so $(X, \mathcal{U}_5)$ is $\kappa$-resolvable (i.e., maximally resolvable) and extraresolvable. Each set $K \in [X'_0]^{<\kappa}$ is closed and discrete in $(X'_0, \mathcal{U}_{0,5}) = (X'_0, \mathcal{U}_5)$, so $\mathrm{nwd}(X'_0, \mathcal{U}_5) = \kappa$. Thus any family of sets dense in $(X, \mathcal{U}_5)$ witnessing the strong extraresolvability of $(X, \mathcal{U}_5)$ would trace on $(X'_0, \mathcal{U}_5)$ to a family witnessing strong extraresolvability there. $\square$

# 5 Some Questions

Both our result cited from [8] in Remark 3.4(b) (where $S(X) > |X|$) and its sequel in Theorem 3.3(b) (where $S(X) \leq |X|$) show that in some cases $\omega$-resolvability suffices to guarantee $\tau$-resolvability for many larger $\tau$. Our methods appear insufficiently delicate, however, to respond to the following question.

**Question 5.1** Let $(X, \mathcal{T})$ be an $\omega$-resolvable Tychonoff space such that $S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T})$. Must $(X, \mathcal{T})$ be $\tau$-resolvable for every $\tau < S(X, \mathcal{T})$?

**Question 5.2** Let $X = (X, \mathcal{T})$ be a dense, $\omega$-resolvable subspace of the space $(D(\kappa))^{2^\kappa}$ such that $|X| = \Delta(X) = \kappa$. [Then $S(X) = \kappa^+$, and $X$ is $\kappa$-resolvable, i.e., maximally resolvable, according to our result [8](4.2).] Does $X$ admit a Tychonoff refinement $\mathcal{U}$ (necessarily with $S(X, \mathcal{U}) = \kappa^+$) such that $\Delta(X, \mathcal{U}) = \Delta(X, \mathcal{T})$, and $(X, \mathcal{U})$ is $\omega$-resolvable but not maximally resolvable? Always? Sometimes? Never?

**Remarks 5.3** (a) Theorem 3.1 sheds no light on Question 5.2, since the hypothesis $S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T})$ is lacking.

(b) The expansion $\mathcal{U}$ of $\mathcal{T}$ requested in Question 5.2, if it exists, cannot be of the kind constructed in this paper. More specifically: There can be no family $\mathcal{W} \subseteq \mathcal{P}(X)$ such that (i) $|U \cap W| = \kappa$ for each $W \in \mathcal{W}$ and $\emptyset \neq U \in \mathcal{T}$, (ii) $\mathcal{U}$ is the smallest topology on $X$ containing $\mathcal{T}$ and $\mathcal{W}$, and (c) each $W \in \mathcal{W}$ is $\mathcal{U}$-clopen. For according to the argument outlined in Discussion 1.5, a space $(X, \mathcal{U})$ arising in that way will embed as a dense subspace of $(D(\kappa))^I$ (with $|I| = w(X, \mathcal{U})$), hence if $\omega$-resolvable is necessarily $\kappa$-resolvable.

(c) Many additional questions relating to (ir)resolvability, together with extensive bibliographic citations, are recorded in the "Problems" article of Pavlov [30].

**Remark 5.4** The reader will have no difficulty using the methods of this paper to establish the following result:

(*) *Let $(X, \mathcal{T})$ be a crowded, maximally resolvable Tychonoff space with $S(X, \mathcal{T}) \leq \Delta(X, \mathcal{T}) = \kappa$. Then for (fixed) $n < \omega$ there is a Tychonoff expansion $\mathcal{U}$ of $\mathcal{T}$ such that $(X, \mathcal{U})$ is $n$-resolvable but not $(n + 1)$-resolvable.*

(Indeed, reducing as in Theorem 4.2 to the case $|X| = \Delta(X, \mathcal{T})$, it is enough to begin with a dense partition $\{D^k_\eta : k < n, \eta < \kappa\}$ of $(X, \mathcal{T})$, a strong small-set-separating $\kappa$-independent family $\mathcal{I} = \{\mathcal{I}_t : t \in X \times 2^\kappa\}$ of $\kappa$ with each $\mathcal{I}_t = \{I^0_t, I^1_t\}$, and with the family $\mathcal{K} = \{K_\xi : \xi < 2^\kappa\}$ defined as in Theorem 2.11. Then the relation $X = \bigcup_{k<n} D^k$ with $D^k := \bigcup_{\eta<\kappa} D^k_\eta$ expresses $(X, \mathcal{U})$ with $\mathcal{U} := \mathcal{T}_{\mathcal{KID}}$ as the union of $n$-many disjoint dense sets, each hereditarily irresolvable by Lemma 2.11(c). A space $(X, \mathcal{U})$ with such a partition cannot be $(n + 1)$-resolvable [23], [10].

We omit the details here of a proof of statement (*) because a stronger Theorem is available, as follows.

(**) *For every $0 < n < \omega$, every $n$-resolvable Tychonoff space $(X, \mathcal{T})$ admits a Tychonoff expansion $\mathcal{U}$ such that $(X, \mathcal{U})$ is $n$-resolvable but not $(n + 1)$-resolvable.*

We will prove (**) in a manuscript now in preparation [9], [10]. We remark *en passant* that *ad hoc* constructions of Tychonoff spaces which for fixed $n < \omega$ are $n$-resolvable but not $(n + 1)$-resolvable have been available for some time [13]; see also [15], [19], [14] and [18] for other examples, not all Tychonoff.

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